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AUTHOR Herriot, Sarah T.; And Others
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ABSTRACT

This course is intended for students who have a thorough knowledge of college preparatory mathematics, including algebra, axiomatic geometry, trigonometry, and analytic geometry. This teacher's guide is for Part II of the course. It is designed to follow Part I of the text. The guide contains background information, suggested instructional materials, and answers to student exercises. (RH)

SCHOOL
MATHEMATICS
STUDY GROUP

# CALCULUS OF ELEMENTARY FUNCTIONS

## Part II

*Teacher's Commentary*

(Revised Edition)

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## Part II

## Teacher's Commentary

(Revised Edition)

The following is a list of all those who participated in the preparation of this volume:


| | |
|---|---|
| Sarah T. Herriot | Gunn High School, Palo Alto, Calif. |
| Desmond T. Jenkins | Palo Alto Senior High School, Calif. |
| C. W. Leeds, III | Simon's Rock, Great Barrington, Mass. |
| George P. Richardson | SMSG, Stanford University, Calif. |
| Donald E. Richmond | Williams College (Emeritus), Williamstown, Mass. |
| Paul C. Shields | Wayne State University, Detroit, Mich. |

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Chapter 6

# DERIVATIVES OF EXPONENTIAL AND RELATED FUNCTIONS

Without the strength of the integral theorems of Chapter 7, it is difficult to frame precise definitions of the power, exponential and logarithmic functions. Our foundation began in Chapter 5 by building intuitively upon familiar algebraic properties (e.g., the n-th power of $a$ means the product of $n$ factors of $a$). Relying on the student's basic concept of power, we were able to discuss exponential and related functions. This leads us now to the study of rates of change of these functions.

We begin discussing the slope of the exponential function in a familiar manner. The slope of the tangent to the graph is initially examined at the y-axis. We note that the slope of any other point is proportional to the value of the function at that point; the constant of proportionality is discovered to be the slope of the graph of the function at the y-axis. The function $x \to e^x$ is defined as that special exponential function for which the constant of proportionality (slope at $(0,1)$) is $1$.

We use plausible arguments to establish the fact that the graph of the function $x \to e^x$ is everywhere convex. (A strict proof would require a continuity argument such as those given in Appendix 7.) We rely heavily on such intuitive geometric arguments as the folding process in order to find the derivative of the inverse of a function whose derivative we already know.

We use such a folding process to find the derivative of the logarithmic function, after the derivative of the exponential function has first been discovered. We use the same mechanical process to obtain the derivative of $x \to \sqrt{x}$, relying on the student's experience with $x \to x^2$ in Chapter 2.

We hope that the student is curious enough to know whether the power rule (which worked so nicely for positive integer exponents in Chapter 2) works for such functions as $x \to x^{1/2}$, $x \to x^{-3/2}$, and $x \to x^{\sqrt{2}}$. In Section 6-6 we extend the power formula to include all real exponents.

Our earlier procedure of first considering the behavior of a graph at y-axis cannot be followed for Taylor approximations of such functions as $x \to \sqrt{x}$ and $x \to \log_e x$. Instead we consider translations of these functions in Section 6-7.

In Example 6-7a we say, "to guarantee accuracy to within 0.005 we could use (9) to show that we must choose $n$ to be at least 199." We illustrate here, using

$$(5) \qquad |R_n| \leq \frac{x^{n+1}}{n+1}, \text{ with } x \geq 0.$$

Since we want to estimate $\log_e 2$ and

$$R_n = \log_e (1 + x) - p(x),$$

we let $x = 1$. We require that

$$|R_n| \leq \frac{1^{n+1}}{n+1} < 0.005.$$

Therefore, we must have

$$\frac{1}{n+1} < 0.005,$$

whence

$$n + 1 > \frac{1000}{5}$$

$$n > 200 - 1$$

$$n > 199.$$

Example 6-7c is intended to use the function $x \to \sqrt{1 + x}$ in order to illustrate assertion (8). There are, of course, methods by which one can obtain more accurate estimates of $\sqrt{\frac{3}{2}}$ more quickly (e.g., guess, divide, and average).

Throughout Chapters 5 and 6 we refer to the function $x \to e^x$ as <u>the</u> exponential function as distinguished from all other exponential functions. We hope that (while we began with base 2) the student realizes that exponential functions with bases other than $e$ are infrequently used. Furthermore, any exponential function is easily expressed in terms of <u>the</u> exponential function by

$$a^x = e^{cx}, \text{ where } c = \log_e a.$$

Similarly, the function $x \to \log_e x$ is referred to in this text (and most advanced books) as the logarithmic function. The student should be told that in other texts (and on some examinations) the function $x \to \log_e x$ is written as $x \to \log x$ or $x \to \ell n\, x$.

Since a logarithm with any base $a$ is simply proportional to the logarithm with base $e$,

$$\log_a x = \frac{\log_e x}{\log_e a}$$

we have

$$D \log_a x = \frac{c}{x}, \quad \text{where} \quad c = \frac{1}{\log_e a}.$$

It may be conjectured that for this reason logarithms with base $e$ are often called "natural" logarithms, natural in the sense that the choice $c = 1$ yields the simplest possible expression for the derivative.

## Solutions Exercises 6-1

1. (a) $f : x \to 8^x = 2^{3x}$. $\therefore$ $\alpha = 3$; $m = 3k \approx 3(.693) \approx 2$.

$f : x \to (\frac{1}{8})^x = 2^{-3x}$. $\therefore$ $\alpha = -3$; $m = -3k \approx -3(.693) \approx -2$

$f : x \to (\sqrt{8})^x = 2^{3/2x}$. $\therefore$ $\alpha = \frac{3}{2}$; $m = \frac{3}{2}k \approx \frac{3}{2}(.693) \approx 1$

$f : x \to (\frac{1}{\sqrt{8}})^x = 2^{-3/2x}$. $\therefore$ $\alpha = -\frac{3}{2}$; $m = -\frac{3}{2}k \approx -\frac{3}{2}(.693) \approx -1$.

(b) The equation of the tangent at $(0,1)$ to the graph of

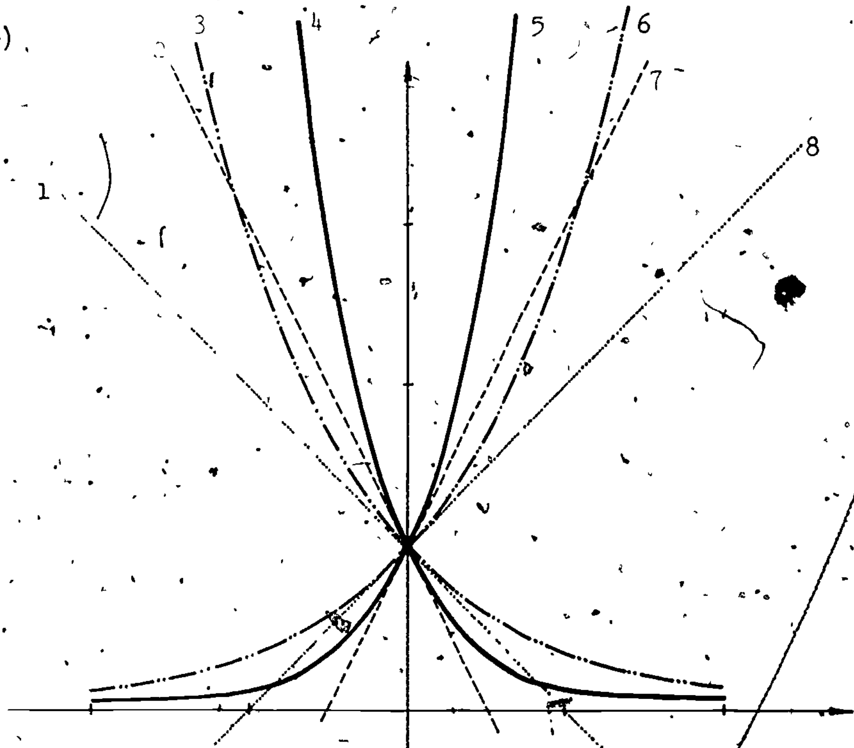
$f : x \to 8^x$ is $y = 1 + 3kx$ or $y \approx 1 + 2x$

$f : x \to (\frac{1}{8})^x$ is $y = 1 - 3kx$ or $y \approx 1 - 2x$

$f : x \to (\sqrt{8})^x$ is $y = 1 + \frac{3}{2}kx$ or $y \approx 1 + x$

$f : x \to (\frac{1}{\sqrt{8}})^x$ is $y = 1 - \frac{3}{2}kx$ or $y \approx 1 - x$

(c)




1. $y = 1 - \frac{3}{2}kx$
2. $y = 1 - 3kx$
3. $y = 8^{-1/2\,x}$
4. $y = (\frac{1}{8})^x$
5. $y = 8^x$
6. $y = 8^{1/2\,x}$
7. $y = 1 + 3kx$
8. $y = 1 + \frac{3}{2}kx$

2. (a) (i) $(1.8)^5 \approx (2^{.85})^5 = \underline{2^{4.25}}$

(ii) $2^{4.25} = (2^4)(2^{.25}) \approx 16(1.189) \approx \underline{19.0}$

(b) (i) $(1.8)^5 \approx (e^{.59})^5 = \underline{e^{2.95}}$ (Interpolation used.)

(ii) $e^{2.95} \approx (e^2)(e^{.95}) \approx (7.389)(2.586) \approx \underline{19.1}$

3. (a) (i) $(0.9)^5 \approx (2^{-.15})^5 = \underline{2^{-.75}}$

(ii) $2^{-.75} \approx .59460 \approx \underline{0.6^-}$

(b) (i) $(0.9)^5 \approx (e^{-.10})^5 = \underline{e^{-.5}}$

(ii) $e^{-.5} \approx \underline{0.6^+}$

4. (a) (i) $(1.02)^8 \approx (2^{.03})^8 = \underline{2^{.24}}$

(ii) $2^{.24} \approx (2^{.20})(2^{.04}) \approx (1.14870)(1.02811) \approx \underline{1.2^-}$

(b) (i) $(1.02)^8 \approx (e^{.02})^8 = \underline{e^{.16}}$

(ii) $e^{.16} = (e^{.15})(e^{.01}) \approx (1.1618)(1.0101) \approx \underline{1.2^-}$

5. (a) $2^{.01} < 1.01 < 2^{.02}$

$2^{.014} < 1.01 < 2^{.015}$

$2^{1.4} < (1.01)^{100} < 2^{1.5}$

(b) $(2)(2^{.4}) < (1.01)^{100} < (2)(2^{.5})$

$2(1.31951) < (1.01)^{100} < 2(1.41421)$

$\therefore\ 2.64 < (1.01)^{100} < 2.83$

Interpolation

$2^{.01} = 1.00696$

$2^{?} = 1.01000$ 304

$2^{.02} = 1.01396$ 700

$(\frac{304}{700})(10) \approx 4.3$

6. $e^{-.70} < 0.5 < e^{-.69}$

$(e^{-.70})^{-12} > (0.5)^{-12} > (e^{-.69})^{-12}$

$4447 > (0.5)^{-12} > 3944$

7. (a) $m_0 = \dfrac{e^h - 1}{h}$

(b) $1 = \dfrac{e^h - 1}{h}$

$h = e^h - 1$

$e^h = 1 + h$

$e = (1 + h)^{1/h}$

(c) $e \approx (1 + .01)^{100}$

$$e \approx 1^{100} + 100 \times 1^{99} \times (.01) + \frac{100 \times 99}{2} \times 1^{98} \times (.01)^2 + \frac{100 \times 99 \times 98}{2 \times 3} \times 1^{97} \times (.01)^3 \approx 2.5$$

(d) For the function $x \to c^x$ we let the slope of the tangent to the graph at $(0,1)$ equal 1. That is, we let

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Using a process similar to that used in part (b) we obtain

$$e = \lim_{h \to 0} (1 + h)^{1/h}.$$

If we let $h = \frac{1}{n}$, we have

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n.$$

8. We assume that $x \to (1 + x)^{1/x}$ is a decreasing function. Without being conscious of this assumption, a student should be ready to believe that $\lim_{h \to 0} (1 + h)^{1/h} > (1 + h)^{1/h}$ for $h > 0$. We let $h = \frac{1}{1000}$, so that

$$(1 + \frac{1}{1000})^{1000} < \lim_{h \to 0} (1 + h)^{1/h}.$$

From Number 7 we know that

$$\lim_{h \to 0} (1 + h)^{1/h} = e.$$

Thus $(\frac{1001}{1000})^{1000} < e$. Even the roughest approximation to $e$ suggests that $2 < e < 3$ and certainly $e < 1000$. Therefore, we can write

$$(\frac{1001}{1000})^{1000} < e < 1000.$$

It follows that

$$(\frac{1001}{1000})^{1000} < 1000.$$

Multiplying by $1000^{1000}$ we get

$$1001^{1000} < 1000^{1001}.$$

The larger of the two numbers is $1000^{1001}$.

9. $f : x \to e^{3x}$; $A(0,e^0)$, $B(h,e^{3h})$. Slope of $AB = \frac{e^{3h} - e^0}{h - 0} = \frac{e^{3h} - 1}{h}$

| $h$ | $3h$ | $e^{3h}$ | $e^{3h} - 1$ | $\frac{e^{3h} - 1}{h}$ |
|---|---|---|---|---|
| .20 | .60 | 1.8221 | 0.8221 | 4.111 |
| .15 | .45 | 1.5683 | 0.5683 | 3.789 |
| .10 | .30 | 1.3499 | 0.3499 | 3.499 |
| .05 | .15 | 1.1618 | 0.1618 | 3.24 |
| .01 | .03 | 1.0305 | 0.0305 | 3.05 |
| .006 | .02 | 1.0202 | 0.0202 | 3.03 |

10. $f : x \to e^{x/2}$; $A(0,e^0)$, $B(h,e^{h/2})$. $m(AB) = \frac{e^{h/2} - e^0}{h - 0} = \frac{e^{h/2} - 1}{h}$

| $h$ | $\frac{h}{2}$ | $e^{h/2}$ | $e^{h/2} - 1$ | $\frac{e^{h/2} - 1}{h}$ |
|---|---|---|---|---|
| .50 | .25 | 1.2840 | 0.2840 | 0.568 |
| .40 | .20 | 1.2214 | 0.2214 | 0.553 |
| .30 | .15 | 1.1618 | 0.1618 | 0.539 |
| .20 | .10 | 1.1052 | 0.1052 | 0.526 |
| .10 | .05 | 1.0513 | 0.0513 | 0.513 |
| .06 | .03 | 1.0305 | 0.0305 | 0.509 |
| .02 | .01 | 1.0101 | 0.0101 | 0.505 |

11. $f : x \to e^{-2x}$; $A(0,e^0)$, $B(h,e^{-2h})$. $m(AB) = \frac{e^{-2h} - e^0}{h - 0} = \frac{e^{-2h} - 1}{h}$

| $h$ | $2h$ | $e^{-2h}$ | $e^{-2h} - 1$ | $\frac{e^{-2h} - 1}{h}$ |
|---|---|---|---|---|
| .20 | .40 | 0.6703 | -0.3297 | -1.649 |
| .15 | .30 | 0.7408 | -0.2592 | -1.726 |
| .10 | .20 | 0.8187 | -0.1813 | -1.813 |
| .05 | .10 | 0.9048 | -0.0952 | -1.90 |
| .02 | .04 | 0.9608 | -0.0392 | -1.96 |
| .01 | .02 | 0.9802 | -0.0198 | -1.98 |
| .005 | .01 | 0.9901 | -0.0099 | -1.98$^+$ |

12. $f : x \to 8^x$; $A(0,8^0)$, $B(h,8^h)$. Therefore, the slope of AB is

$$\frac{8^h - 8^0}{h - 0} = \frac{2^{3h} - 1}{h},$$

i.e., $f : x \to 2^{3x}$.

| $h$ | $3h$ | $2^{3h}$ | $2^{3h} - 1$ | $\frac{2^{3h} - 1}{h}$ |
|---|---|---|---|---|
| .20 | .60 | 1.51572 | 0.51572 | 2.5786 |
| .15 | .45 | 1.36604 | 0.36604 | 2.4403 |
| .10 | .30 | 1.23114 | 0.23114 | 2.3114 |
| .05 | .15 | 1.10957 | 0.10957 | 2.191 |
| .01 | .03 | 1.02101 | 0.02101 | 2.101 |
| .006 | .02 | 1.01396 | 0.01396 | 2.094 |
| .0006 | .002 | 1.0013865 | 0.0013865 | 2.07972 |
| -.20 | -.60 | 0.64302 | -.35698 | 1.7849 |
| -.15 | -.45 | 0.73204 | -.26796 | 1.3398 |
| -.10 | -.30 | 0.81225 | -.18775 | 1.8775 |
| -.05 | -.15 | 0.90125 | -.09875 | 1.975 |
| -.01 | -.03 | 0.97942 | -.02058 | 2.058 |
| -.006 | -.02 | 0.98623 | -.01377 | 2.065 |
| -.0006 | -.002 | 0.9986147 | -.0013853 | 2.07798 |

From this table where $h$ approaches zero from the right, and also from the left, we see that

as $h \to 0$, $2.07798 < \frac{2^{3h} - 1}{h} < 2.07972$

or as $h \to 0$, $2.078 < \frac{2^{3h} - 1}{h} < 2.080$.

Solutions Exercises 6-2

1. (a) $m = e^{-1} \approx 0.36788$

   (b) $m = e^{0.5} \approx 1.6487$

   (c) $m = e^{0.7} \approx 2.0138$

   (d) $m = 1$

   (e) $m = e^{1.5} \approx 4.4817$

2. See graph.

3. (a) $y = e^{-1}(x + 1) + e^{-1} = e^{-1}(x + 2)$

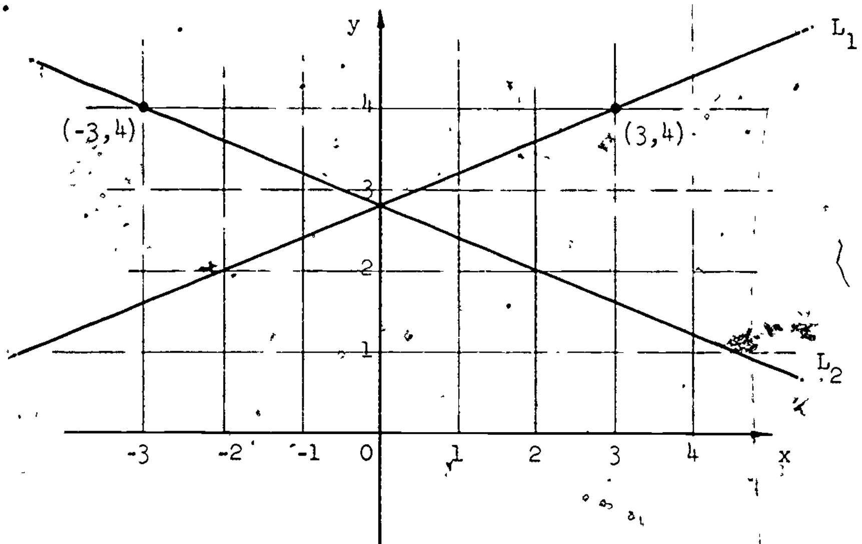
   (b) $y = e^{0.5}(x - .5) + e^{0.5} = e^{0.5}(x + .5)$

   (c) $y = e^{0.7}(x - .7) + e^{0.7} = e^{0.7}(x + .3)$

   (d) $y = x + 1$

   (e) $y = e^{1.5}(x - 1.5) + e^{1.5} = e^{1.5}(x - .5)$

4.

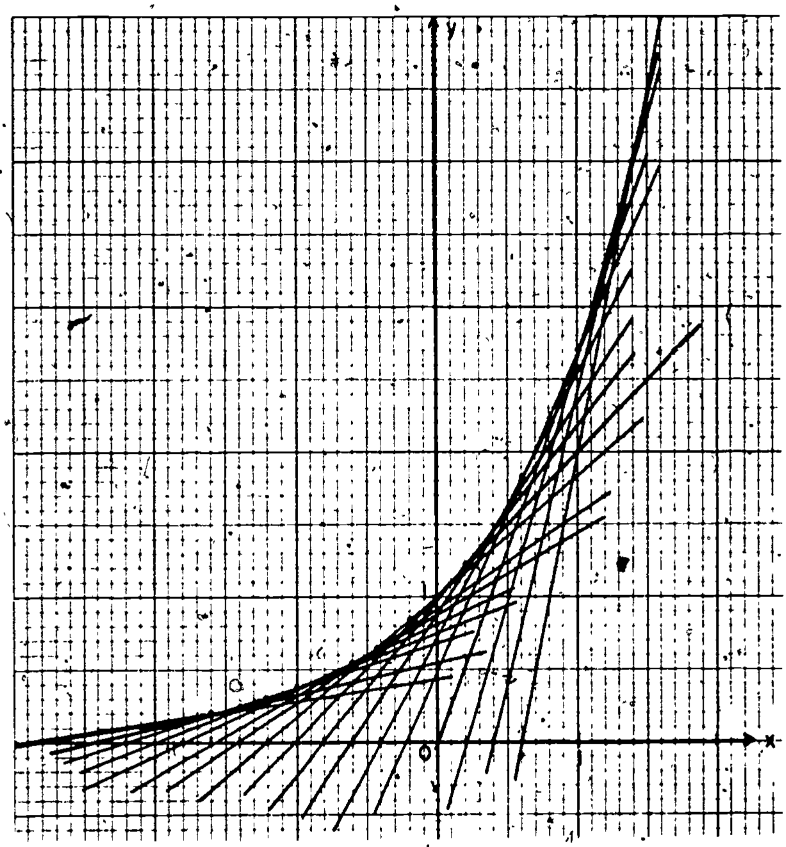



   (c) Point (-3,4)

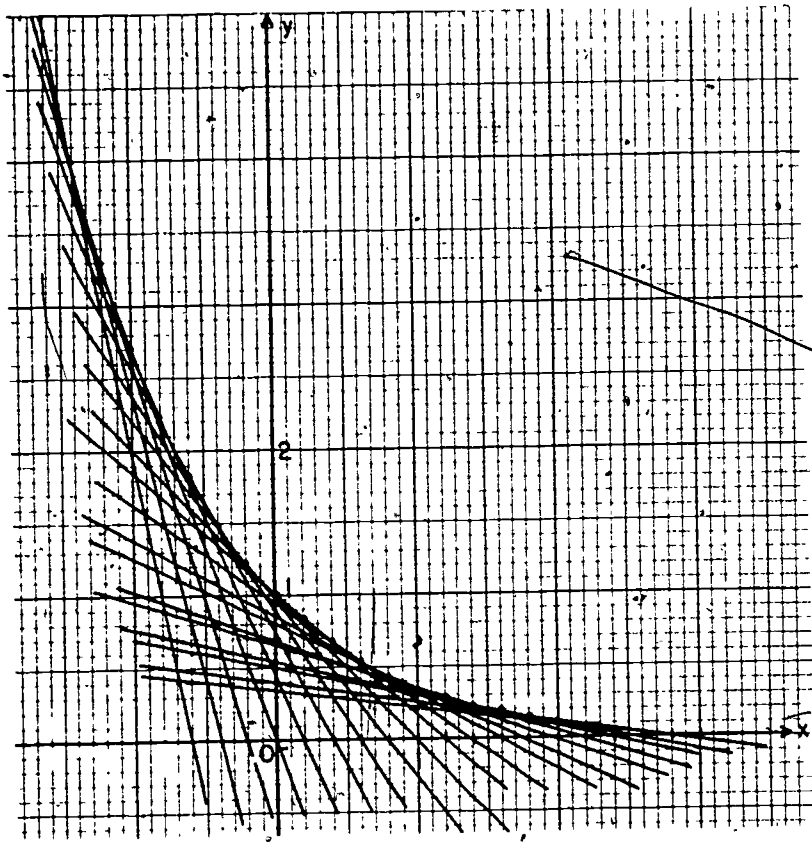
   (d) The slope of $L_2$ is $-\frac{2}{5}$.

   (e) Point $(-r,s)$ on $L_2$ corresponds to point $(r,s)$ on $L_1$. The slope of $L_2$ is $-m$.

5. This exercise is starred because of the time needed to complete it. The figure should be shown to the class even if the exercise is not assigned.



6. (a)




(b) Each point has coordinates $(-x, e^{-x})$.

(c) The slope of each line drawn in 7(a) is the negative of the slope of the corresponding line drawn in 6(b).

7. (a) See Number 2 for the graph of $f : x \to e^{x}$. The graph of $g : x \to e^{-x}$ may be obtained by reflecting the graph of $f$ in the y-axis.

(b)

| | Slope of graph of $f$ | Slope of graph of $g$ |
|---|---|---|
| at $x = 0$ | 1 | -1 |
| at $x = +1$ | $e \approx 2.72$ | $-\frac{1}{e} \approx -0.37$ |
| at $x = -1$ | $\frac{1}{e} \approx 0.37$ | $-e \approx -2.72$ |

(c) At $x = h$ the slope of the graph of $g : x \to e^{-x}$ is

$$-e^{-h} = -g(h).$$

Solutions Exercises 6-3

1. $A = Pe^{rt} = 1000\ e^{(.03)(18)} = 1000\ e^{0.54}$

   $e^{0.54} \approx 1.7160$

   $A \approx 1000(1.7160) \approx 1716$

   Mr. Toffey will have approximately \$1,716 for Jack's education on Jack's eighteenth birthday. (Perhaps he can apply for a scholarship.)

2. $A = Pe^{rt}$

   For $r = .03$ we require that $A = 2P$.

   $2P = Pe^{0.03t}$

   $2 = e^{0.03t}$

   $2 \approx e^{.693}$

   $e^{0.693} \approx e^{0.03t}$

   $.03t \approx .693$

   $t \approx 23.1$

   It takes approximately 23 years.

3. Jack Toffey will be 23 years old.

4. We could use a table of natural logarithms, but we shall use $2 \approx e^{0.693}$ as we did in Number 2.

   (a) $r = 0.06$

   $2P = Pe^{0.06t}$

   $e^{0.06t} = 2 \approx e^{0.693}$

   $t \approx 11.5$

   The time required is approximately 12 years.

5. (a) If the number of interest periods were $n$ we would have for the principal at the end of one year

$$P_n = P_0\left(1 + \frac{.0485}{n}\right)^n$$

where $P_0$ is the initial deposit. The principal at the end of a year on the basis of continuous compounding is

$$P = \lim_{n \to \infty} P_n = P_0\left[\lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^{.0485z}\right] \qquad \left(z = \frac{n}{.0485}\right)$$

$$= P_0 \left[\lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z\right]^{.0485}$$

$$= P_0 e^{.0485}$$

$$= P_0\left(1 + .0485 + \frac{(.0485)^2}{2!} + \frac{(.0485)^3}{3!} + \ldots\right).$$

To three terms we have

$$P = P_0(1.0497),$$

equivalent to simple interest of 4.97%.

(b) In $x$ years, the principal accumulated is $P_0e^{.0485x}$. We require $P = 2P_0$, hence $x = \frac{\log_e 2}{.0485} \approx \frac{0.693}{.0485} \approx 14.3$. It takes approximately 14 years.

6. Since $P = 180$ and $P_0 = 760$, $180 = 760e^{-0.11445h}$, and $e^{-0.1144h} = \frac{180}{760} \approx 0.237$. From the Tables we see that $0.237 \approx e^{-1.45}$; hence $-0.1144h \approx -1.45$, and $h \approx 12.7$. Thus the height is about 12.7 kilometers.

7. $Q = \frac{Q_0}{2} = Q_0e^{-0.12n}$ so that $\frac{1}{2} = e^{-0.12n}$. Since $0.5 \approx e^{-0.69}$, $-0.12n \approx -0.69$ and $n \approx 5\frac{3}{4}$. The required time is about $5\frac{3}{4}$ days.

8. $I(5) = I_0 e^{-5k} = \frac{2}{3} I_0$, so that $e^{-5k} = \frac{2}{3}$.

$I(10) = I_0 e^{-10k} = I_0 (e^{-5k})^2 = \frac{4}{9} I_0$.

Hence, 10 feet below the surface the intensity is $\frac{4}{9} I_0$.

[As an alternative method, we note that

$$\frac{2}{3} = \frac{I(5)}{I(0)} = \frac{I(5 + 5)}{I(0 + 5)} = \frac{I(10)}{I(5)}.$$

It follows that

$$I(5) = \frac{2}{3} I_0 \text{ and } I(10) = \frac{2}{3} I(5) = (\tfrac{2}{3})^2 I_0.]$$

If $I(x) = \frac{1}{2} I_0$, $\frac{1}{2} = e^{-kx}$. From above, $e^{-5k} = \frac{2}{3} \approx e^{-0.40}$, so that $k \approx 0.08$. Thus $e^{-0.08x} \approx 0.5 \approx e^{-0.70}$ and $x \approx 8.75$; hence at depth of about $8\frac{3}{4}$ feet the intensity is $\frac{1}{2} I_0$.

Solutions Exercises 6-4

1. (a) $\log_e(1.96) = \log_e(1.4)^2 = 2\log_e(1.4) = 2(0.3365) = \underline{0.6730}$

(b) $\log_e(2.03) = \log_e(2.9)(0.7) = \log_e(2.9) + \log_e(0.7) = (1.0647) + (-.3567)$

$= \underline{0.7080}$

(c) (i) $\log_e(0.52) = \log_e(\frac{3.9}{7.5})$

$= \log_e(3.9) - \log_e(7.5) = (1.3610) - (2.0149) = \underline{-0.6539}$

(ii) $\log_e(0.52) = \log_e(\frac{5.2}{10})$

$= \log_e(5.2) - \log_e(10) = (1.6487) - (2.3026) = \underline{-0.6539}$

(d) $\log_e(0.052) = \log_e \frac{5.2}{10^2} = \log_e(5.2) - 2\log_e(10) = (1.6487) - (4.6052)$

$= \underline{-2.9565}$

(e) $\log_e(\frac{750,000}{39,000,000}) = \log_e \frac{(7.5)(10^5)}{(3.9)(10^7)} = \log_e(\frac{7.5}{3.9}) + \log_e(\frac{1}{10^2})$.

[See No. 1(c)(i) and No. 1(d).] $= +0.6539 - 4.6052 = \underline{-3.9513}$

2. (a) $\log_e\sqrt{2} = \frac{1}{2}\log_e 2 \approx \frac{1}{2}(0.6931) \approx 0.3466$. $\therefore$ $\underline{\sqrt{2} \approx 1.41}$

(b) $\log_e \sqrt[3]{71} = \frac{1}{3}[\log_e(7.1) + \log_e(10)] \approx \frac{1.9601 + 2.3026}{3} \approx 1.4209$.

$\therefore$ $\underline{\sqrt[3]{71} \approx 4.14}$

(c) $\log_e(9.1)^{2/3} = \frac{2}{3}\log_e(9.1) \approx \frac{2(2.2083)}{3} \approx 1.4722$ $\therefore$ $\underline{(9.1)^{2/3} \approx 4.36}$

(d) $\log_e(100)^{1/e} = \frac{2}{e}\log_e 10$

$$\left.\begin{aligned}
\log_e[\log_e(100)^{1/e}] &= \log_e 2 - \log_e e + \log_e[\log_e 10] \\
&\approx .6931 - 1 + \log_e(2.3026) \\
&\approx .6931 - 1 + .8340 \\
&\approx .5271 \\
\therefore\ \log_e(100)^{1/e} \approx 1.694 &\text{ and } \underline{(100)^{1/e} \approx 5.44}
\end{aligned}\right\}$$

or alternately:

$$\left\{\begin{aligned}
\log_e(100)^{1/e} &= 2(e^{-1})\log_e 10 \\
&\approx 2(.3679)(2.3026) \\
&\approx 1.694 \\
\therefore\ \underline{(100)^{1/e} \approx 5.44}
\end{aligned}\right.$$

3. For some $x$ close to $c$, we have by (5)

$$\log_e x \approx \log_e c + \frac{1}{c}(x - c).$$

In all parts of this problem $c = 2$, $\frac{1}{c} = 0.5$, and $\log_e 2 = .6931$.

(a) $\log_e(2.01) \approx .6931 + (.5)(.01)$ $\qquad x = 2.01$

$\approx .6931 + .005$ $\qquad x - c = 0.01$

$\approx .6981$

(b) $\log_e(1.96) \approx .6931 + (.5)(-.04)$ $\qquad x = 1.96$

$\approx .6931 - .020$ $\qquad x - c = -.04$

$\approx .6731$

(c) $\log_e(2.03) \approx .6931 + (.5)(.03)$ $\qquad x = 2.03$

$\approx .6931 + .015$ $\qquad x - c = .03$

$\approx .7081$

(d) $\log_e(1.94) \approx .6931 + (.5)(-.06)$ $\qquad x = 1.94$

$\approx .6931 - .030$ $\qquad x - c = -.06$

$\approx .6631$

4. (a) Use result of Number 3(a) to find $(2.01)^{5/3}$

$\log(2.01)^{5/3} = \frac{5}{3}\log(2.01) \approx \frac{5}{3}(.6981) \approx 1.1635$ $\quad \therefore (2.01)^{5/3} \approx 3.20$

(b) Use result of Number 3(b) to find $\sqrt[6]{1.96}$

$\log\sqrt[6]{1.96} = \frac{1}{6}\log(1.96) \approx \frac{1}{6}(.6731) \approx .1122$ $\quad \therefore \sqrt[6]{1.96} \approx 1.12$

(c) Use result of Number 3(c) to find $(2.03)^{.6}$

$\log(2.03)^{.6} = (.6)\log(2.03) \approx (.6)(.7081) \approx (.42486)$

$\therefore (2.03)^{.6} \approx 1.53$

(d) Use result of Number 3(d) to find $(1.94)^{(1.1)}$

$\log(1.94)^{(1.1)} = (1.1)\log(1.94) \approx (1.1)(.6631) \approx .72941$

$\therefore (1.94)^{1.1} \approx 2.07$

5\. (a) (i) When $y = 0$, $\log_e 3x = 0$, and $3x = 1$.

Therefore the x-intercept is at the point $(\frac{1}{3}, 0)$

(ii) $\log_e 2x = 0$, and $2x = 1$. $\therefore$ x-intercept: $(\frac{1}{2}, 0)$

(iii) $\log_e x = 0$, and $x = 1$. $\therefore$ x-intercept: $(1,0)$

(iv) $\log_e \frac{x}{2} = 0$, and $x = 2$. $\therefore$ x-intercept: $(2,0)$

(v) $\log_e \frac{x}{3} = 0$, and $x = 3$. $\therefore$ x-intercept: $(3,0)$

(vi) $\log_e \frac{x}{4} = 0$, and $x = 4$. $\therefore$ x-intercept: $(4,0)$

(b) (i) If $\log_e kx = 0$, then $kx = 1$, and $x = \frac{1}{k}$.

Since $k > 1$, then $x$ is in the interval $0 < x < 1$.

(ii) $\lim_{k \to \infty} x = \lim_{k \to \infty} \frac{1}{k} \to 0$.

(c) (i) If $\log_e \frac{x}{k} = 0$, then $\frac{x}{k} = 1$, and $x = k$.

Since $k > 1$, then $x > 1$.

(ii) $\lim_{k \to \infty} x = \lim_{k \to \infty} k \to \infty$.

6\. (a) Find the difference (positive) between each logarithm.

(i) $\log_e 2x - \log_e x = \log_e \frac{2x}{x} = \log_e 2 = \log_e (1 + \frac{1}{1})$

(ii) $\log_e 3x - \log_e 2x = \log_e \frac{3x}{2x} = \log_e \frac{3}{2} = \log_e (1 + \frac{1}{2})$

(iii) $\log_e 4x - \log_e 3x = \log_e \frac{4x}{3x} = \log_e \frac{4}{3} = \log_e (1 + \frac{1}{3})$

(iv) $\log_e (k + 1)x - \log_e kx = \log_e \frac{k+1}{k} = \log_e (1 + \frac{1}{k})$

(b) $\lim_{k \to \infty} [\log_e(k + 1)x - \log_e(kx)] = \lim_{k \to \infty} \log_e(1 + \frac{1}{k}) \to \log_e 1 = \underline{0}$

7\. (a) (Similar to No. 6.) Find the difference (positive) between each logarithm.

(i) $\log_e x - \log_e \frac{x}{2} = \log_e 2$

(ii) $\log_e \frac{x}{2} - \log_e \frac{x}{3} = \log_e \frac{3}{2}$

(iii) $\log_e \frac{x}{3} - \log_e \frac{x}{4} = \log_e \frac{4}{3}$

(iv) $\log_e \frac{x}{k} - \log_e \frac{x}{k+1} = \log_e \frac{k+1}{k}$

(b) $\lim_{k \to \infty} [\log_e(\frac{x}{k}) - \log_e (\frac{x}{k+1})] = \lim_{k \to \infty} \log_e(\frac{k+1}{k})$

$= \lim_{k \to \infty} \log_e(1 + \frac{1}{k}) \to \log_e 1 = \underline{0}$

8. (a) (i) If $f : x \to \log_e 2x$, then $f : x \to (\log_e 2 + \log_e x)$.

It follows that $f' : x \to \frac{1}{x}$

(ii) Since $\log_e \frac{x}{2} = \log_e x - \log_e 2$, $f' : x \to \frac{1}{x}$

(iii) Since $\log_e 3x = \log_e 3 + \log_e x$, $f' : x \to \frac{1}{x}$

(iv) Since $\log_e \frac{x}{3} = \log_e x - \log_e 3$, $f' : x \to \frac{1}{x}$

(v) Since $\log_e kx = \log_e k + \log_e x$, $f' : x \to \frac{1}{x}$ k(constant) > 0

(vi) Since $\log_e \frac{x}{k} = \log_e x - \log_e k$, $f' : x \to \frac{1}{x}$ k(constant) > 0

(b) $f'(e) = \frac{1}{e}$ for each of the curves in part (a) above.

(c) (i) $f(e) = \log_e 2e$ $\therefore$ The coordinates are $(e, \log_e 2e)$ or $(e, 1 + \log_e 2)$

(ii) $f(e) = \log_e \frac{e}{2}$ $\therefore$ The coordinates are $(e, \log_e \frac{e}{2})$ or $(e, 1 - \log_e 2)$

(iii) $f(e) = \log_e 3e$ $\therefore$ The coordinates are $(e, \log_e 3e)$ or $(e, 1 + \log_e 3)$

(iv) $f(e) = \log_e \frac{e}{3}$ $\therefore$ The coordinates are $(e, \log_e \frac{e}{3})$ or $(e, 1 - \log_e 3)$

(v) $f(e) = \log_e ke$ $\therefore$ The coordinates are $(e, \log_e ke)$ or $(e, 1 + \log_e k)$

(vi) $f(e) = \log_e \frac{e}{k}$ $\therefore$ The coordinates are $(e, \log_e \frac{e}{k})$ or $(e, 1 - \log_e k)$

(d) (i) $y = 1 + \log_e 2 + \frac{1}{e}(x - e) = \frac{1}{e} x + \log_e 2$

(ii) $y = 1 - \log_e 2 + \frac{1}{e}(x - e) = \frac{1}{e} x - \log_e 2$

(iii) $y = 1 + \log_e 3 + \frac{1}{e}(x - e) = \frac{1}{e} x + \log_e 3$

(iv) $y = 1 - \log_e 3 + \frac{1}{e}(x - e) = \frac{1}{e} x - \log_e 3$

(v) $y = 1 + \log_e k + \frac{1}{e}(x - e) = \frac{1}{e} x + \log_e k$

(vi) $y = 1 - \log_e k + \frac{1}{e}(x - e) = \frac{1}{e} x - \log_e k$

(e) (i) The tangents (listed in Solution to No. 8(d) above) cross the y-axis at the following points, respectively.

(i) $\log_e 2$

(ii) $-\log_e 2$ or

$\log_e \frac{1}{2}$

(iii) $\log_e 3$

(iv) $-\log_e 3$ or

$\log_e \frac{1}{3}$

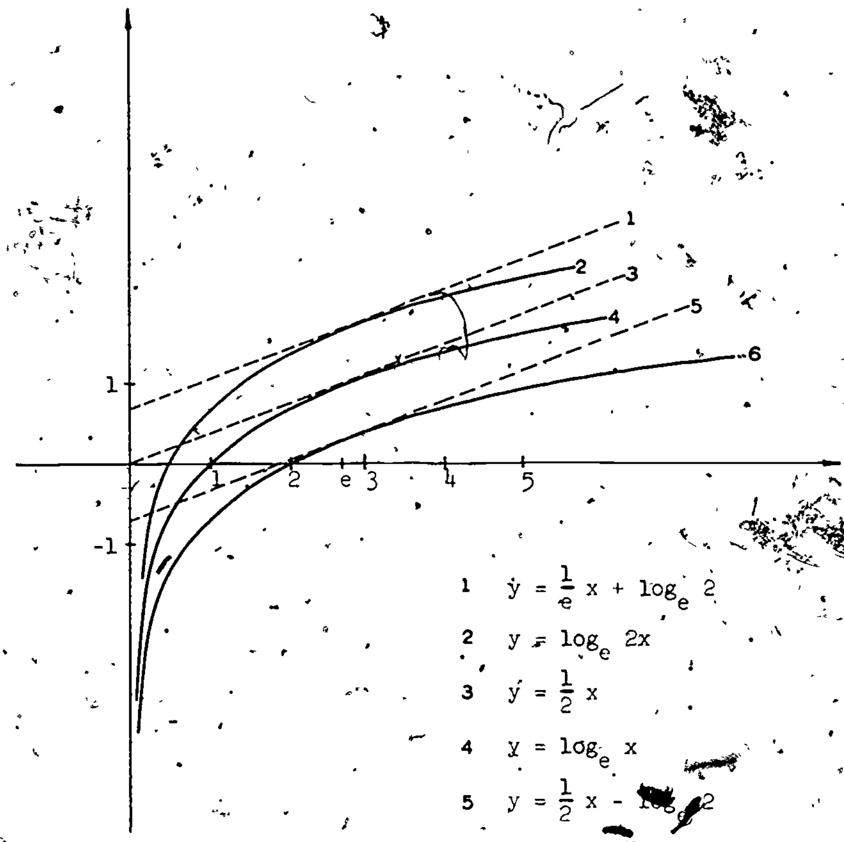
(v) $\log_e k$

(vi) $-\log_e k$ or

$\log_e \frac{1}{k}$

(ii) See part (d), (v) and (vi): $\log_e k$ and $-\log_e k$ are symmetric with respect to the origin.

(f)




1 $y = \frac{1}{e} x + \log_e 2$

2 $y = \log_e 2x$

3 $y = \frac{1}{2} x$

4 $y = \log_e x$

5 $y = \frac{1}{2} x - \log_e 2$

6 $y = \log_2 \frac{x}{2}$

9. (a) (i) $D(\log_e x^2) = D(2 \log_e x) = \frac{2}{x}$

(ii) $D(\log_e x^3) = D(3 \log_e x) = \frac{3}{x}$

(iii) $D \log_e \sqrt{x} = D(\frac{1}{2} \log_e x) = \frac{1}{2x}$

(iv) $D \log_e \sqrt[3]{x} = D(\frac{1}{3} \log_e x) = \frac{1}{3x}$

(b) (i) $D(\log_e x^n) = D(n \log_e x) = \frac{n}{x}$

(ii) $D(\log_e \sqrt[n]{x}) = D(\frac{1}{n} \log_e x) = \frac{1}{nx}$

(iii) $D(\log_e (cx + d)^n) = D[n \log_e (cx + d)] = \frac{nc}{cx + d}$

(iv) $D \log_e \sqrt[3]{x} = D(\frac{1}{3} \log_e x) = \frac{1}{3x}$

10. (a) $f(x) = \log_e (5x + 1)^3 = 3 \log_e (5x + 1)$

$\therefore f'(x) = \frac{3 \cdot 5}{5x + 1} = \frac{15}{5x + 1}$

(b) $f(x) = \log_e (4x^2 \cdot \sqrt{x}) = \log_e (4x^{5/2}) = \log_e 4 + \frac{5}{2} \log_e x$

$\therefore f'(x) = 0 + \frac{5}{2} \cdot \frac{1}{x} = \frac{5}{2x}$

(c) $f(x) = \log_e x(1 - 2x) = \log_e x + \log_e (1 - 2x)$

$\therefore f'(x) = \frac{1}{x} - \frac{2}{1 - 2x} = \frac{1 - 4x}{x(1 - 2x)}$

(d) $f(x) = \log_e x^2(3x - 1) = 2 \log_e x + \log_e (3x - 1)$

$\therefore f'(x) = \frac{2}{x} + \frac{3}{3x - 1} = \frac{9x - 2}{x(3x - 1)}$

(e) $f(x) = \log_e [\log_e e^x] = \log_e x$

$f'(x) = \frac{1}{x}$

(f) $f(x) = \log_e (\sin \frac{\pi}{2})$

$f'(x) = 0$ since $\log_e (\sin \frac{\pi}{2})$ is a constant.

(g) $f(x) = \log_e \frac{2x - 1}{2x + 1} = \log_e (2x - 1) - \log_e (2x + 1)$

$\therefore f'(x) = \frac{2}{2x - 1} - \frac{2}{2x + 1} = \frac{2(2x + 1 - 2x + 1)}{(2x - 1)(2x + 1)} = \frac{4}{4x^2 - 1}$

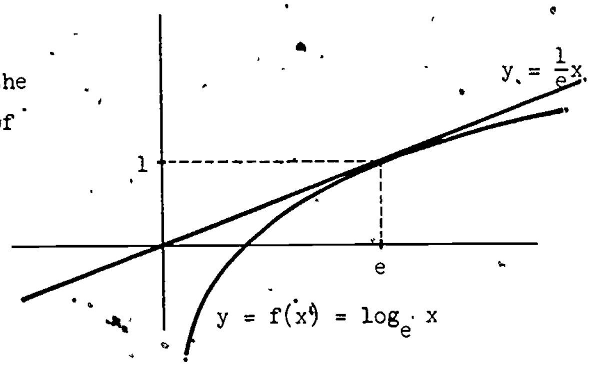
(h) $f(x) = \log_e \sqrt{\frac{1 + x}{1 - x}} = \frac{1}{2}[\log_e (1 + x) - \log_e (1 - x)]$

$\therefore f'(x) = \frac{1}{2}[\frac{1}{1 + x} - \frac{-1}{1 - x}] = \frac{1}{1 - x^2}$

11. The equation of the tangent line is of the form $y = mx + b$. Since the line is tangent to the graph of $f : x \to \log_e x$, its slope is $m = f'(x) = \frac{1}{x}$ for some value of $x > 0$. Since the line is required to pass through the origin we have $b = 0$. At the point of tangency we must have $y = \frac{1}{x} \cdot x + 0 = 1$, for $x > 0$. If $y = 1$ at the point of tangency then $\log_e x = 1$; that is, $x = e^1$. Thus, $m = f'(e) = \frac{1}{e}$, and (we already said) $b = 0$.

Therefore the equation of the only tangent to the graph of $y = \log_e x$ that passes through the origin is

$$y = \frac{1}{e} x.$$




In Example 6-4a we concluded that the equation of the tangent was

$$y = 1 + \frac{1}{e}(x - e).$$

We can simplify this to agree with the result of this problem:

$$y = 1 + \frac{1}{e} x - 1.$$

$$y = \frac{1}{e} x.$$

## Solutions Exercises 6-5

1. (a) $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

   (b) $-e^x \approx -1 - x - \frac{x^2}{2!} - \frac{x^3}{3!}$

   (c) $1 - e^x \approx - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!}$

   (d) $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

   (e) $-\cos x \approx -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}$

   (f) $1 - \cos x \approx \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}$

2. $f : x \to y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$

   (a) $y \approx a_0 + a_1x$

   (b) $y \approx a_0 + a_1x + a_2x^2$

   (c) $y \approx a_0 + a_1x + a_2x^2 + a_3x^3$

3. $g : x \to y = \sin x$

   (a) $y \approx x$

   (b) $y \approx x$

   (c) $y \approx x - \frac{x^3}{3!}$

4. $F : x \to y = \cos x$

   (a) $y \approx 1$

   (b) $y \approx 1 - \frac{x^2}{2!}$

   (c) $y \approx 1 - \frac{x^2}{2!}$

5. $G : x \to y = e^x$

   (a) $e^x \approx 1 + x$

   (b) $e^x \approx 1 + x + \frac{x^2}{2!}$

   (c) $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

6. No. The function $f : x \to \log_e x$ is not defined at $x = 0$.

7. $e^{0.01} \approx 1 + 0.01 + \frac{(0.01)^2}{2!} + \frac{(0.01)^3}{3!}$

$\approx 1.01 + .00005 + 0.000000167$

$\approx 1.010050$

We needed to use only 3 terms, since $\frac{(0.01)^n}{n!}$ contains zero in the first six places for $n \geq 3$.

8. $e^{0.1} \approx 1.105$

$e^{0.2} = (e^{0.1})^2 \approx 1.221$

$e^{0.4} = (e^{0.1})^4 \approx 1.4908$

$e^{0.8} = (e^{0.1})^8 \approx 2.2225$

$e = (e^{0.2})(e^{0.8}) \approx (1.2210)(2.2225) \approx 2.714$

9. (a) $p_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \frac{x^n}{n!}$.

$p_n'(x) = 1x^0 + \frac{2x}{2!} + \frac{3x^2}{3!} + \dots + \frac{nx^{n-1}}{n!}$

Since $\frac{k}{k!} = \frac{1}{(k-1)!}$ then

$p_n'(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!}$.

$p_n'(x) + \frac{x^n}{n!} = p_n(x)$

$p_n'(x) = p(x) - \frac{x^n}{n!}$

(b) From (a) $p_n'(x) = p_n(x) - \frac{x^n}{n!}$. Since $\frac{x^n}{n!} > 0$ if $x > 0$ we have

$$p_n'(x) < p_n(x) \quad \text{for} \quad x > 0.$$

(c) $p_n'(x) < p_n(x)$ for $x > 0$ and $f'(x) = f(x)$ for $f : x \to e^x$, $p_n(0) = f(0)$.

The intuitive geometric point is stressed in this problem. Observe that both functions have the same value at $x = 0$. But $p'(x) < p(x) \approx f(x) = f'(x)$ thus $p'(x) < f'(x)$. This means that $f$ is rising more rapidly than $p$ and $p(x) < f(x)$ for $x > 0$. (Use 6-4-(4) with $\geq$ signs if you wish a more exacting approach.)

10. (a) $g(x) - p_3(x) = (c - 1)\frac{x^3}{3!}$

Since $c > 1$ and $\frac{x^3}{3!} > 0$ if $x > 0$ this gives

$g(x) - p_3(x) > 0$ for $x > 0$.

(b) $g'(x) = 1x^0 + \frac{2x^1}{2!} + \frac{3cx^2}{3!} = 1 + x + \frac{cx^2}{2!}$ so that

$$g'(x) - g(x) = (1 + x + \frac{cx^2}{2!}) - (1 + x + \frac{x^2}{2!} + \frac{cx^3}{3!})$$

$$= (c - 1)\frac{x^2}{2!} - \frac{cx^3}{3!}$$

$$= \frac{x^2}{2!}((c - 1) - \frac{cx}{3})$$

which will be positive if $x > 0$ and $(c - 1) - \frac{cx}{3} > 0$ that is $x < 3(\frac{c - 1}{c})$.

(c) (i) If $0 < 2 < \frac{3(c - 1)}{c}$ then

$$2c < 3c - 3$$

and $$3 < c.$$

The smallest integer value is $c = 4$.

(ii) $f(2) = e^2 \approx 7.3891$ from tables. When $c = 4$

$$g(2) = 1 + \frac{2}{1} + \frac{2^2}{2!} + \frac{4 \cdot 2^3}{3!}$$

$$= 1 + 2 + 2 + \frac{16}{3}$$

$$= 10.\overline{3}$$

Thus $g(2) > f(2)$.

(d) We will appeal to an intuitive argument. Since $f(0) = g(0)$, the two functions have the same initial values when $x$ is close enough to zero, or as $x \to 0^+$, $g(x) \approx f(x)$. We find that $f'(x)$ is equal to $f(x)$; the slope and the function have the same numerical values for a given $x$. But $g'(x) > g(x)$ when $x > 0$, from part (b). Thus $g'(x) > g(x) \approx f(x) = f'(x)$ and $g'(x) > f'(x)$ near $x = 0$. Graphically, $g(x)$ and $f(x)$ start out the same near zero. Very quickly $g(x)$ has a steeper slope than $f(x)$ which leads us to the conclusion that $g(x) > f(x)$ for $0 < x < \frac{3(c - 1)}{c}$.

11. The fact that $p_n(x) < e^x$ for $x > 0$ was proved in Number 9. Calculation gives

$$g_n'(x) = 1 \cdot x^0 + \frac{2x}{2!} + \frac{3x^2}{3!} + \ldots + \frac{cn\ x^{n-1}}{n!}$$

$$= 1 + x + \frac{x^2}{2!} + \ldots + \frac{cx^{n-1}}{(n - 1)!}$$

so that

$$g_n'(x) - g(x) = (c - 1)\frac{x^{n-1}}{(n - 1)!} - \frac{cx^n}{n!}$$

$$= \frac{x^{n-1}}{(n - 1)!}\left(c - 1 - \frac{cx}{n}\right)$$

which will be positive for $0 < x < n(\frac{c - 1}{c})$. Arguing as in Number 10(d) we have $e^x < g_n(x)$ in the stated interval.

Thus $p_n(x) < e^x \leq g_n(x)$ if $0 < x < \frac{n(c - 1)}{c}$.

12. $p_n(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} < e^x < g_n(x)$, $0 < x < n(\frac{c - 1}{c})$ where

$g_n(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{cx^n}{n!}$. Subtract $p_n(x)$ to obtain

$$0 < e^x - p_n(x) < g_n(x) - p_n(x) = \frac{(c - 1)x^n}{n!}.$$

13. Using (5) we have

$$|R_n| \leq \frac{e^2|x|^{n+1}}{(n + 1)!} \text{ if } |x| \leq 2.$$

Replacing $e^2$ by 9 and $|x|$ by 2 we have

$$|R_n| < \frac{9 \times 2^{n+1}}{(n + 1)!}.$$

To obtain two decimal place accuracy we need to know that

$$|R_n| < .005$$

so it is enough to choose the first $n$ for which

$$\frac{9 \times 2^{n+1}}{(n + 1)!} < .005,$$

that is

$$\frac{2^{n+1}}{(n + 1)!} < \frac{.005}{9} = .00055\ldots < .00064$$

so it is enough to choose $n$ so that

$$\frac{2^{n+1}}{(n+1)!} < 2^6 \times 10^{-5}\,,$$

that is

$$\frac{2^{n-5}}{(n+1)!} < 10^{-5} = \frac{1}{100,000}\,.$$

Try

$$n = 6: \quad \frac{2^{6-5}}{7!} = \frac{2}{5040} = \frac{1}{2520}$$

$$n = 7: \quad \frac{2^{7-5}}{8!} = \frac{4}{5040 \times 8} = \frac{1}{10,080}$$

$$n = 8: \quad \frac{2^{8-5}}{9!} = \frac{8}{5040 \times 8 \times 9} = \frac{1}{45,360}$$

$$n = 9: \quad \frac{2^{9-5}}{10!} = \frac{1 \times 2}{45,360 \times 10} = \frac{2}{453,600}$$

$$= \frac{1}{226,800} < \frac{1}{100,000}$$

so $n = 9$ will work!

14. We have

$$|R_n| \le \frac{e^{0.5}(\frac{1}{2})^{n+1}}{(n+1)!} < \frac{2(\frac{1}{2})^{n+1}}{(n+1)!} \quad \text{if} \quad |x| \le 0.5.$$

It is enough to choose $n$ so that

$$\frac{1}{(n+1)!2^n} < .00005 = \frac{5}{100,000} = \frac{1}{20,000}\,.$$

$$n = 3: \quad \frac{1}{4!\,2^3} = \frac{1}{24 \times 8} = \frac{1}{192}\,,$$

$$n = 4: \quad \frac{1}{5!\,2^4} = \frac{1}{192 \times 5 \times 2} = \frac{1}{1,920}$$

$$n = 5: \quad \frac{1}{6!\,2^6} = \frac{1}{1920 \times 6 \times 2} = \frac{1}{23,040} < \frac{1}{20,000}$$

so we choose $n = 5$.

15. We have

$$|R_n| \le \frac{e^{0.001}(\frac{1}{1000})^{n+1}}{(n+1)!} \quad \text{if} \quad |x| \le 0.001$$

with $n = 1$ and using $e^{0.001} < 2$ we have

$$|R_n| \le \frac{2 \times 1000^{-2}}{2!} = 10^{-6} < 5 \times 10^{-6}$$

so correct to 5 decimal places

$$e^{0.001} = 1 + 0.001 = 1.00100$$

$$e^{-0.001} = 1 - 0.001 = .99900.$$

16. (a) $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + R_5$

where

$$|R_5| \le \frac{e^M|x|^6}{6!} \quad \text{if} \quad |x| \le M.$$

Thus

$$e^{cx} \approx 1 + cx + \frac{(cx)^2}{2!} + \dots + \frac{(cx)^5}{5!} + R_5$$

where

$$|R_5| \le \frac{e^M|cx|^6}{6!} \quad \text{if} \quad |cx| \le M.$$

(b) $e^{x^2} \approx 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + R$

where

$$|R| \le \frac{e^M x^{10}}{5!} \quad \text{if} \quad |x| \le \sqrt{M}.$$

17. (a) $e^x = 1 + x + R_1$ where $|R_1| \le \frac{|x|^2}{2!}$ if $|x| \le 1$

$e^{-x^2} = 1 - x^2 + R_1, \qquad |R_1| \le \frac{|x|^4}{2!}$ if $|x| \le 1$

$\therefore\ 1 - e^{-x^2} = x^2 - R_1$

$\sin x = x + R_2$ where $|R_2| \le \frac{|x|^3}{3!}$

so $$\frac{(1 - e^{-x^2})\sin x}{x^3} = \frac{x^3 - xR_1 + x^2R_2 - R_1R_2}{x^3}$$

which approaches 1 since

$$\left|\frac{-xR_1 + x^2R_2 - R_1R_2}{x^3}\right| \le \frac{\frac{|x|^5}{2} + \frac{|x|^5}{6} + \frac{|x|^7}{12}}{|x|^3}$$

$$= \frac{|x|^2}{2} + \frac{|x|^2}{6} + \frac{|x|^4}{12} \to 0$$

as $x \to 0$.

(b) $e^x = 1 + x + R_1$, $|R_1| \le \dfrac{|x|^2}{2!}$, $|x| \le 1$

$\cos x = 1 + R_2$, $|R_2| \le \dfrac{|x|^2}{2!}$

so that

$$\frac{e^x - \cos x}{x} = \frac{x + R_1 - R_2}{x} = 1 + \frac{R_1 - R_2}{x}$$

which approaches 1 since

$$\left|\frac{R_1 - R_2}{x}\right| \le \frac{\frac{|x|^2}{2} + \frac{|x|^2}{2}}{x} = |x| \to 0 \quad \text{as} \quad x \to 0.$$

(c) $$\frac{\cos x^2 - e^{x}}{\sin x^3} = \frac{\cos x^2 - e^{x^4}}{x^3} \cdot \frac{x^3}{\sin x^3},$$

we have

$$\cos x^2 \approx 1 - \frac{x^4}{2!}, \quad e^{x^4} \approx 1 + x^4$$

so

$$\frac{\cos x^2 - e^{x^4}}{x^2} \approx \frac{-3x}{2} \to 0 \quad \text{as} \quad x \to 0.$$

Since $\dfrac{x^3}{\sin x^3} \to 1$, the correct result is 0.

You may wish the students to use error estimates to prove this (as in (a) and (b)).

18. $e - e^x = e(1 - e^{x-1})$

$e^x \approx 1 + x$ so $e^{x-1} \approx 1 + (x - 1)$ and hence

$$\frac{e(1 - e^{x-1})}{x - 1} \approx -\frac{e(x - 1)}{x - 1} = -e,$$

so $$\lim_{x \to 1} \frac{e - e^x}{x - 1} = -e.$$

You may wish the students to use error estimates as in Number 11(a), (b).

## Solutions Exercises 6-6

1\. (a) $f : x \to 2x^{3/2}$

$f' : x \to 3x^{1/2}$

(b) $f : x \to \frac{6}{\sqrt{x}} = 6 \cdot x^{-1/2}$

$f' : x \to -3x^{-3/2} = \frac{-3}{|x|\sqrt{x}}$

(c) $f : x \to \frac{5}{2} x^{2/5}$

$f' : x \to x^{-3/5} = \frac{1}{\sqrt[5]{x^3}}$

(d) $f : x \to (\frac{x}{10})^{1/10}$

$f' : x \to \frac{1}{10} \cdot \frac{1}{10^{1/10}} \cdot \frac{1}{x^{9/10}} = \frac{1}{10^{11/10}\ x^{9/10}}$

(e) $f : x \to \sqrt{2x}$

$f' : x \to 2 \cdot \frac{1}{2}(2x)^{-1/2} = \frac{1}{\sqrt{2x}}$

(f) $f : x \to \frac{4}{\sqrt[3]{8x^2}} = 2x^{-2/3}$

$f' : x \to -\frac{4}{3} x^{-5/3} = \frac{-4}{3x\ \sqrt[3]{x^2}}$

(g) $f : x \to \frac{1}{2}\sqrt{\frac{1}{2x}} = 2^{-1}(2x)^{-1/2}$

$f' : x \to 2^{-1}(-\frac{1}{2})(2)(2x)^{-3/2} = -\frac{1}{2(2x)^{3/2}}$

$\therefore\ f' : x \to \frac{-1}{4|x|\sqrt{2x}}$

(h) $f : x \to 20(\frac{3x}{\pi})^{.7}$

$f' : x \to 14(\frac{3}{\pi})(\frac{3x}{\pi})^{(-.3)}$

$\therefore\ f' : x \to \frac{42}{\pi}(\frac{\pi}{3x})^{(.3)}.$

(i) $f : x \to 2\ \frac{\sqrt[3]{x}}{\sqrt{2x}} = \sqrt{2}\ x^{-1/6}$

$f' : x \to \frac{-\sqrt{2}}{6} x^{-7/6} = \frac{-1}{3\sqrt{2}\ |x|\ \sqrt[6]{x}}$

(j) $f : x \to \frac{4}{3} \cdot \frac{1}{x} = \frac{4}{3} x^{-1}$

$f' : x \to -\frac{4}{3}x^{-2} = \frac{-4}{3x^2}$

2. f is defined, respectively, for the following values of x:

(a) $x \geq 0$
(b) $x > 0$
(c) All $x$
(d) $x \geq 0$
(e) $x \geq 0$
(f) $x \neq 0$
(g) $x > 0$
(h) $x \geq 0$
(i) $x > 0$
(j) $x \neq 0$

3. f is defined, respectively, for the following values of x:

(a) $x \geq 0$
(b) $x > 0$
(c) $x \neq 0$
(d) $x > 0$
(e) $x > 0$
(f) $x \neq 0$
(g) $x > 0$
(h) $x > 0$
(i) $x > 0$
(j) $x \neq 0$

4. (a) $f'(x) = 3x^{1/2}$

$f'(1) = 3$

$f'(2) = 3\sqrt{2}$

(b) $f'(x) = \dfrac{-3}{|x|\sqrt{x}}$

$f'(1) = -3$

$f'(2) = \dfrac{-3}{2\sqrt{2}}$

(c) $f'(x) = \dfrac{1}{\sqrt[5]{x^3}}$

$f'(1) = 1$

$f'(2) = \dfrac{1}{\sqrt[5]{8}}$

(d) $f'(x) = \dfrac{1}{10^{11/10}\, x^{9/10}}$

$f'(1) = \dfrac{1}{10^{11/10}}$

$f'(2) = \dfrac{1}{10^{11/10}\, 2^{9/10}}$

(e) $f'(x) = \dfrac{1}{\sqrt{2x}}$

$f'(1) = \dfrac{1}{\sqrt{2}}$

$f'(2) = \dfrac{1}{2}$

(f) $f'(x) = \dfrac{-4}{3x\sqrt[5]{x^2}}$

$f'(1) = -\dfrac{4}{3}$

$f'(2) = \dfrac{-4}{3 \cdot 2\sqrt[5]{2^2}} = \dfrac{-4}{6\sqrt[5]{4}}$

(g) $f'(x) = \dfrac{-1}{4|x|\sqrt{2x}}$

$f'(1) = \dfrac{-1}{4\sqrt{2}}$

$f'(2) = \dfrac{-1}{16}$

(h) $f'(x) = \dfrac{42}{\pi}\left(\dfrac{\pi}{3x}\right)^{(.3)}$

$f'(1) = \left(\dfrac{42}{\pi}\right)\left(\dfrac{\pi}{3}\right)^{(.3)} \doteq 14\left(\dfrac{3}{\pi}\right)^{(.7)}$

$f'(2) = \dfrac{42}{\pi}\left(\dfrac{\pi}{6}\right)^{(.3)}$

(i) $f'(x) = \dfrac{-1}{3\sqrt{2}\,|x|\,\sqrt[6]{x}}$

$f'(1) = -\dfrac{1}{3\sqrt{2}}$

$f'(2) = \dfrac{-1}{6\sqrt{2}\,\sqrt[6]{2}} = \dfrac{-1}{6\sqrt[3]{4}}$

(j) $f'(x) = \dfrac{-4}{3x^2}$

$f'(1) = -\dfrac{4}{3}$

$f'(2) = -\dfrac{1}{3}$

5. (a), (c), (d), (e), (h)

6. (a)

7. (a) $D\sqrt{x+1} = D(x+1)^{1/2} = \frac{1}{2}(x+1)^{-1/2} = \dfrac{1}{2\sqrt{x+1}}$

(b) $D\sqrt[3]{x-4} = D(x-4)^{1/3} = \frac{1}{3}(x-4)^{-2/3} = \dfrac{1}{3\sqrt[3]{(x-4)^2}}$

(c) $D\dfrac{1}{(x+2)^3} = D(x+2)^{-3} = -3(x+2)^{-4} = \dfrac{-3}{(x+2)^4}$

(d) $D\sqrt{\dfrac{4}{x}} = D2x^{-1/2} = 2\cdot(-\frac{1}{2})x^{-3/2} = \dfrac{-1}{\sqrt{x^3}}$

(e) $D\sqrt{2}\cdot\sqrt{x+\frac{3}{2}} = \sqrt{2}\,D(x+\frac{3}{2})^{1/2} = \sqrt{2}\cdot\frac{1}{2}(x+\frac{3}{2})^{-1/2} = \dfrac{\sqrt{2}}{2\sqrt{x+\frac{3}{2}}}$,

$$\frac{1}{\sqrt{2}\sqrt{\frac{2x+3}{2}}} = \frac{1}{\sqrt{2x+3}}$$

(f) $D\dfrac{\sqrt{x-1}}{\sqrt{x^2-1}} = D\sqrt{\dfrac{x-1}{(x-1)(x+1)}} = D(x+1)^{-1/2} = (-\frac{1}{2})(x+1)^{-3/2}$

$$= \frac{-1}{2\sqrt{(x+1)^3}}$$

(g) $D\dfrac{b}{\sqrt{cx+d}} = D\dfrac{b}{\sqrt{c}\sqrt{x+\frac{d}{c}}} = \dfrac{b}{\sqrt{c}}D(x+\frac{d}{c})^{-1/2}$

$$= \frac{b}{\sqrt{c}}(-\frac{1}{2})(x+\frac{d}{c})^{-3/2} = \frac{-b}{2\sqrt{c}\sqrt{(x+\frac{d}{c})^3}}$$

$$= \frac{-b}{2\sqrt{c}\sqrt{\frac{(cx+d)^3}{c^3}}} = \frac{-b}{2\sqrt{\frac{c}{c^3}}\sqrt{(cx+d)^3}} = \frac{-bc}{2\sqrt{(cx+d)^3}}$$

8. The respective functions are defined...

(a) for all $x \geq -1$

(b) for all $x$

(c) for all $x \neq -2$

(d) for all $x > 0$

(e) for all $x \geq -\frac{3}{2}$

(f) for all $x > 1$

(g) for all $x > -\frac{d}{c}$

9. The respective derivatives are defined...

(a) for all $x > -1$

(b) for all $x \neq 4$

(c) for all $x \neq -2$

(d) for all $x > 0$

(e) for all $x > -\frac{3}{2}$.

(f) for all $x \neq -1$

(g) for all $x > -\frac{d}{c}$

10. $f : x \to 2(1 - x)^{1/2}$

(a) $f' : x \to \frac{-1}{\sqrt{1 - x}}$

$f'(-8) = -\frac{1}{3}$

$f'(-3) = -\frac{1}{2}$

Neither $f$ nor $f'$ is defined at $x = 2$.

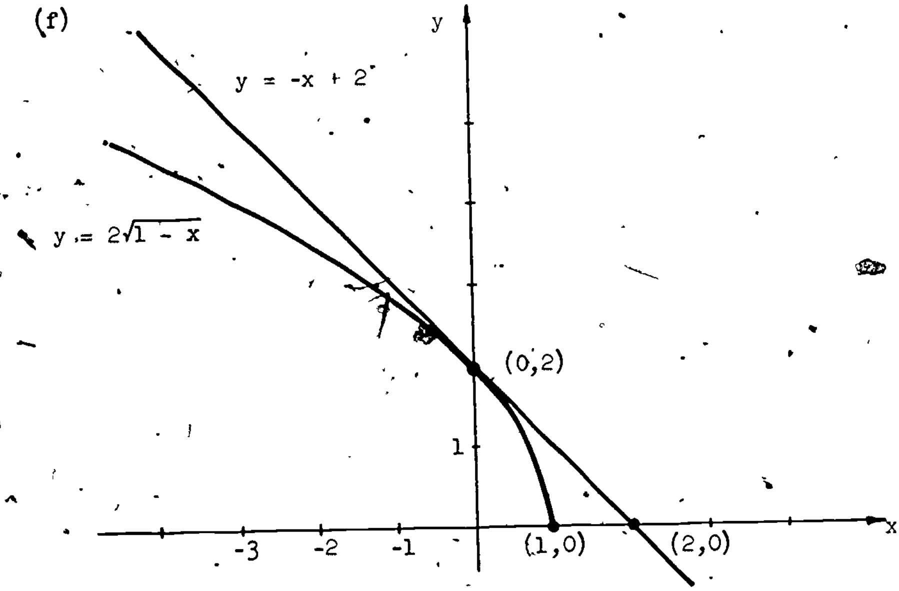
(b) $f$ is the positive branch of a parabola which would have the equation: $y^2 = 1 - x$. $\therefore$ $f$ is defined for $x \leq 1$.

(c) $f'$ is defined for $x < 1$.

(d) Since $f'$ is always negative, the function is decreasing for $x < 1$. Since $f'$ is never positive, the function is never increasing.

(e) $f'(0) = -1$. If $x = 0$, $f(0) = 2$. Therefore the equation of the tangent to the curve at $x = 0$ is $\underline{y = -x + 2}$.

(f)




11. Given $f : x \to \sqrt[3]{x^2}$

(a) $f' : x \to \dfrac{2}{3\sqrt[3]{x}}$

(b) When $f' < 0$, $x < 0$, and $f$ is decreasing.
When $f' > 0$, $x > 0$, and $f$ is increasing.
The endpoint of the interval, $x = 0$, can be considered as part of either or neither interval, as you see fit.

(c) $f(1) = 1$; $f'(1) = \frac{2}{3}$

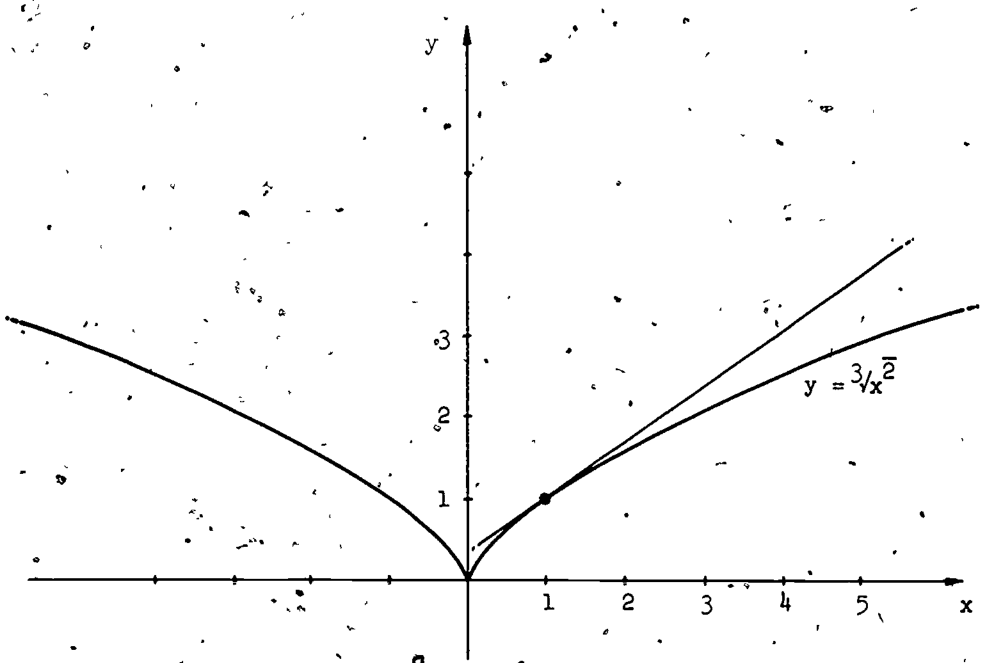
Equation of tangent at $(1,1)$: $y = \frac{2}{3}(x - 1) + 1$ or $y = \frac{1}{3} + \frac{2}{3}x$

(d) Let $f' = -1$; i.e., $\dfrac{2}{3x^{1/3}} = -1$; $x^{1/3} = -\frac{2}{3}$; $x = -\frac{8}{27}$.

$f(-\frac{8}{27}) = \frac{4}{9}$. Therefore the coordinates of the point on the curve where the slope is $-1$ are $(-\frac{8}{27}, \frac{4}{9})$; the equation of the tangent line is $y = -(x + \frac{8}{27}) + \frac{4}{9}$ or $y = \frac{4}{27} - x$.

(e) Yes, it is a vertical line with the slope undefined. The equation is $x = 0$.

(f)




12. Given: $f : x \to x - \frac{1}{x}$

(a) $f' : x \to 1 + \frac{1}{x^2} = \frac{x^2 + 1}{x^2}$

$f' > 0$ $\therefore$ The curve is increasing for all values for which it is defined; i.e., $|x| > 0$. It is never decreasing since $f'$ is never negative.

(b) As $|x|$ increases, the curve approaches $y = x$.

(c) $\frac{x^2 + 1}{x^2} = 5$; $4x^2 = 1$; $x = \frac{1}{2}$ or $x = -\frac{1}{2}$

i.e., at $(\frac{1}{2}, -\frac{3}{2})$ and $(-\frac{1}{2}, \frac{3}{2})$, the slope is 5. Therefore the equations of the tangents are

$$y = 5(x - \tfrac{1}{2}) - \tfrac{3}{2} \quad \text{and} \quad y = 5(x + \tfrac{1}{2}) + \tfrac{3}{2}.$$

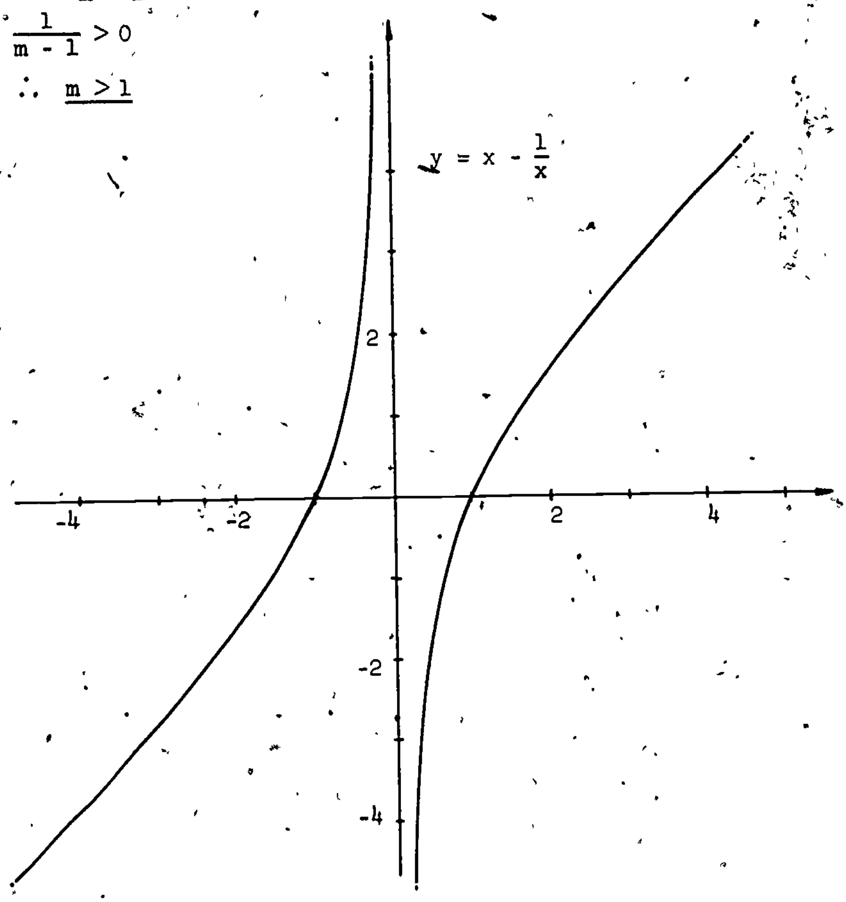
(d) $\frac{x^2 + 1}{x^2} = m$

$x^2 = \frac{1}{m - 1}$

$\frac{1}{m - 1} > 0$

$\therefore$ $\underline{m > 1}$

(e)




13. $p : x \to 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{10}}{10!}$

Here we anticipate the next section. It is not intended that you should teach for this problem.

(a) $p' : x \to 1 + x + \frac{x^2}{2!} + \dots + \frac{x^9}{9!}$

$p'' : x \to 1 + x + \frac{x^2}{2!} + \dots + \frac{x^8}{8!}$

$p''' : x \to 1 + x + \frac{x^2}{2!} + \dots + \frac{x^7}{7!}$

(b) $p(0) = 1$

$p'(0) = 1$

$p''(0) = 1$

$p'''(0) = 1$

(c) The student should guess that $f' = f$.

Solutions Exercises 6-7

1. We have

$$|R_n| \leq \frac{x^{n+1}}{n+1} \leq \frac{1}{n+1}$$

so
$$\frac{1}{n+1} \leq 0.005 = \frac{5}{1000} = \frac{1}{200}$$

gives $n + 1 \geq 200$, that is $n \geq 199$. Secondly

$$\frac{1}{n+1} \leq 5 \times 10^{-10}$$

gives
$$n + 1 \geq \frac{1}{5} \times 10^{10} = 2 \times 10^9$$

so that
$$n \geq 2 \times 10^9 - 1 = 1,999,999,999$$

(a rather large number).

2. With $x = 0.2$ we have (from 9)

$$|R_n| \leq \frac{x^{n+1}}{n+1} = \frac{1}{5^{n+1}(n+1)}$$

so that for

$$n = 2: \quad \frac{1}{5^3 \times 3} = \frac{1}{375} < \frac{1}{200} = 0.005.$$

Hence

$$\log_e 1.2 = \log_e(1 + 0.2) \approx p_2(0.2)$$
$$= (0.2) - \frac{(0.2)^2}{2}$$
$$= 0.2 - 0.02$$
$$= 0.18$$

which is correct to two places.

3. $\log_e(0.1) = \log_e(1 - 0.9)$ with $x = -0.9$.

This time we use (8) since $-1 < x \leq 0$.

$$|R_n| \leq \frac{|x|^{n+1}}{(1+x)(1+n)} < 0.05.$$

It might be helpful to allow each $n + 1$ to be a power of 2. This will help in computation.

$$n = 1,\quad n + 1 = 2,\quad \frac{(0.9)^2}{(0.1)(2)} = \frac{0.81}{.2} \approx 4.05 > 0.05$$

$$n = 3,\quad n + 1 = 4,\quad \frac{(0.9)^4}{(0.1)(4)} = \frac{(0.81)^2}{.4} = \frac{.6561}{.4} \approx 1.64$$

$$n = 7,\quad n + 1 = 8\quad \frac{(0.9)^8}{(0.1)(8)} = \frac{(.6561)^2}{.8} \approx \frac{.4365}{.8} \approx .546$$

$$n = 15,\quad n + 1 = 16,\quad \frac{(0.9)^{16}}{(0.1)16} \approx \frac{(.4365)^2}{1.6} \approx \frac{.1901}{1.6} \approx .119$$

$$n = 31,\quad n + 1 = 32,\quad \frac{(0.9)^{32}}{(0.1)32} \approx \frac{(.1901)^2}{3.2} \approx \frac{.01416}{3.2} \approx .004$$

Then certainly $n = 31$ is large enough. For a more precise $n$ we could use logarithms in our computations.

$$R_{21} \approx 0.0447 < 0.05 < 0.052 \approx R_{20}.$$

Thus $n = 21$ is the smallest $n$ which is large enough.

4. (a) $\log_e 3 = \log_e(1 + 2)$

$$\approx p_5(2) = 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \frac{2^5}{5}$$

$$= 2 - \frac{4}{2} + \frac{8}{3} - \frac{16}{4} + \frac{32}{5}$$

$$= \frac{304}{60} \approx 5.067$$

(b) $|R_5| \leq \frac{2^6}{6} = 10^{2/3}$

(c) Tables give $\log_e 3 \approx 1.0986$

(d) $n = 6$: $\log_e 3 \approx \frac{304}{60} - \frac{2^6}{6} \approx -5.600$

$n = 7$: $\log_e 3 \approx -5.600 + \frac{2^7}{7} \approx 12.686$

$n = 8$: $\log_e 3 \approx 12.686 - \frac{2^8}{8} \approx -19.314$

$n = 9$: $\log_e 3 \approx -19.314 + \frac{2^9}{9} \approx 37.575$

(e) The difference $\log_e 3 - p_n(2)$ oscillates in sign and its absolute value approaches $+\infty$ as $n \to \infty$.

5. (a) $\log_e (1 + x) \approx x$ if $|x|$ is small, so $\frac{\log_e(1 + x)}{x} \approx 1$ and hence limit is 1. A more formal proof, using error estimates is as follows:

$$\log_e(1 + x) = x + R_1 \quad \text{where} \quad |R_1| \leq \frac{|x|^2}{1 + x} \quad \text{if} \quad |x| < 1 .$$

so that

$$\frac{\log_e(1 + x)}{x} = 1 + \frac{R_1}{x} \xrightarrow[x \to 0]{} 1$$

since

$$\left|\frac{R_1}{x}\right| \leq \frac{|x|}{1 + x} \xrightarrow[x \to 0]{} 0.$$

(b)
$$\left.\begin{aligned} \sin x &\approx x \\ 1 - \cos x &\approx \frac{x^2}{2} \\ \log_e(1 + x) &\approx x \end{aligned}\right\} \quad \text{if } |x| \text{ is small}$$

so

$$\lim_{x \to 0} \frac{(\sin x)\log_e(1 + x)}{(1 - \cos x)} = \lim_{x \to 0} \frac{x^2}{\frac{x^2}{2}} = 2$$

You may wish the students to use error estimates as in (a).

6. With $f : x \to \sqrt{1 + x}$, we have

$$f(0) = 1, \quad f'(0) = \frac{1}{2}, \quad f''(0) = -\frac{1}{4},$$

$$f'''(0) = \frac{3}{8}, \quad f^{(4)}(0) = -\frac{15}{16}$$

(as in (10)). Also

$$f^{(4)} : x \to -\frac{15}{16}(1 + x)^{-7/2}$$

so

$$f^{(5)} : x \to \frac{105}{32}(1 + x)^{-9/2}$$

$$f^{(6)} : x \to -\frac{945}{64}(1 + x)^{-11/2}.$$

Hence:

$$f^{(5)}(0) = \frac{105}{32}$$

so (using (12))

$$a_0 = 1, \quad a_1 = \frac{1}{2}, \quad a_2 = -\frac{1}{8}, \quad a_3 = \frac{1}{16},$$

$$a_4 = -\frac{5}{128}, \quad a_5 = \frac{7}{256}, \quad a_6 = -\frac{21}{1024}$$

and

$$p_5(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5$$

with

$$|R_5| \leq |a_6|x^6 = \frac{21}{1024}x^6 \quad \text{if} \quad 0 \leq x \leq 1.$$

7. $n = 4$: $\sqrt{2} = \sqrt{1 + 1}$

$$\approx p_4(1) = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128}$$

$$= \frac{179}{128} \approx 1.399$$

with a maximum error (from (13) with $x = 1$)

$$|R_4| \leq \frac{7}{256} < 0.028$$

$n = 5$: $\sqrt{2} = \sqrt{1 + 1}$

$$\approx p_5(1) = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \frac{7}{256}$$

$$= \frac{179}{128} + \frac{7}{256} = \frac{365}{256} \approx 1.426$$

with

$$|R_5| \leq \frac{21}{1024} < 0.0206.$$

Each gives $\sqrt{2} = 1.4$ correct to one decimal place. More terms are needed to improve accuracy.

8. $n = 3$: $\sqrt{1.1} = \sqrt{1 + 0.1}$

$$\approx p_3(0.1) = 1 + \frac{0.1}{2} - \frac{(0.1)^2}{8} + \frac{(0.1)^3}{16}$$

$$= 1 + 0.05 - 0.00125 + 0.0000625$$

$$= 1.0488125$$

with

$$|R_3| \leq \frac{5}{128} \cdot (0.1)^4 = 0.00000390625.$$

Hence, correct to five decimal places $\sqrt{1.1} = 1.04881$.

$n = 4$: $\sqrt{1.1} = \sqrt{1 + 0.1}$

$$\approx p_4(0.1) = 1 + \frac{0.1}{2} - \frac{(0.1)^2}{8} + \frac{(0.1)^3}{16} - \frac{5(0.1)^4}{128}$$

$$= 1.0488125 - 0.00000390625$$

$$= 1.04880859375$$

with

$$|R_4| \leq \frac{7}{256} (0.1)^5 < 0.0000003.$$

9. $\sqrt{\frac{1}{2}} = \sqrt{1 - 0.5}$

$$\approx p_4(0.5) = 1 + \left(\frac{-0.5}{2}\right) - \frac{(-0.5)^2}{8} + \frac{(-0.5)^3}{16} - \frac{5}{128}(-0.5)^4$$

$$\approx 0.7085$$

which agrees with

$$\sqrt{\frac{1}{2}} \approx 0.707$$

in the first two decimal places.

10. (a)

$$\left.\begin{aligned} \cos x &\approx 1 - \frac{x^2}{2} \\ \sqrt{1+x} &\approx 1 + \frac{x}{2} \\ \log_e(1+x) &\approx x \end{aligned}\right\} \text{ if } |x| \text{ is small}$$

so

$$\frac{\cos x - \sqrt{1+x}}{\log_e(1+x)} \approx \frac{(1 - \frac{x^2}{2}) - (1 + \frac{x}{2})}{x} = \frac{-1}{2} - \frac{x}{2}$$

and hence the limit is $\frac{-1}{2}$. Again you may wish a more vigorous disucssion using error estimates to establish

$$\frac{\cos x - \sqrt{1+x}}{x} \to -\frac{1}{2} \quad \text{and} \quad \frac{\log_e(1+x)}{x} \to 1.$$

(b)

$$\left.\begin{aligned} e^{x^2} &\approx 1 + x^2 \\ \sqrt{1+x^2} &\approx 1 + \frac{x^2}{2} \\ (\sin x)^2 &\approx x^2 \end{aligned}\right\} \text{ if } |x| \text{ is small}$$

so

$$\frac{e^{x^2} - \sqrt{1+x^2}}{(\sin x)^2} \approx \frac{1}{2}.$$

11. (a) Put $f : x \to (1+x)^{1/3}$. Calculate (using (1)):

$$f' : x \to \frac{1}{3}(1+x)^{-2/3}$$

$$f'' : x \to -\frac{2}{9}(1+x)^{-5/3}$$

$$f''' : x \to \frac{10}{27}(1+x)^{-8/3}$$

so that

$$f(0) = 1, \quad f'(0) = \frac{1}{3}, \quad f''(0) = -\frac{2}{9}, \quad f'''(0) = \frac{10}{27}$$

and

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3.$$

(b) Put $f : x \to (1 + x)^{5/3}$

$\therefore\ f' : x \to \frac{5}{3}(1 + x)^{2/3}$

$f'' : x \to \frac{10}{9}(1 + x)^{-1/3}$

$f''' : x \to -\frac{10}{27}(1 + x)^{-4/3}$

Hence

$$f(0) = 1, \quad f'(0) = \frac{5}{3}, \quad f''(0) = \frac{10}{9}, \quad f'''(0) = -\frac{10}{27}$$

$$p_3(x) = 1 + \frac{5}{3}x + \frac{5}{9}x^2 - \frac{5}{81}x^3.$$

# Chapter 7

## AREA AND THE INTEGRAL

Our approach is intuitive as we discuss the following topics in Chapter 7:

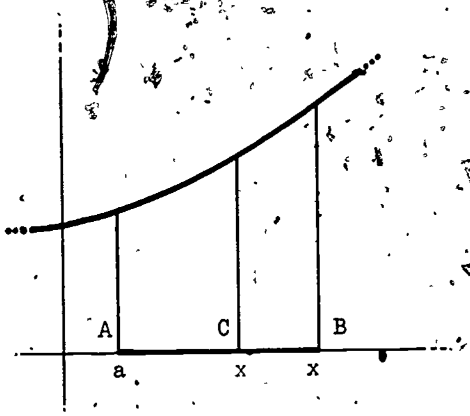
7-1. Area Under a Graph
7-2. The Area Theorem
7-3. The Fundamental Theorem of Calculus
7-4. Properties of Integrals
7-5. Signed Area
7-6. Integration Formulas

Extension and a more analytical approach to the ideas discussed in Chapter 7 can be found in the following sections of the appendices:

A5-2. Evaluation of an Area
A3-2. Sums and Sum Notation
A5-3. Integration by Summation Techniques
A5-4. The Concept of Integral. Integrals of Monotone Functions
A5-5. Elementary Properties of Integrals
A8-1. Existence of Integrals
A8-2. The Integral of a Continuous Function
A9-1. The Logarithm as Integral
A9-2. The Exponential Functions

The following discussion may be helpful to some students as they study Section 7-4 and/or in anticipation of Section 9-5.

One is tempted to write $\int_a^x f(x)\,dx$ for the area below the graph of $f$ and above the segment $\overline{AB}$ (see figure). In fact, this is often done. The difficulty is that then $x$ stands for two different things as we have shown:




(1) the abscissa of the right end $B$ of the interval;

(2) the abscissa of a point like $C$ within the interval.

This is confusing. Consequently, it is wise to use a different letter (say $t$) for one or the other of these abscissas.

It is customary to keep $x$ for the end of the interval and to change the variable under the integral sign to $t$, say, so that

$$\int_a^x f$$

is written as

$$\int_a^x f(t)dt.$$

For example, if $f : x \to x^2$

$$\int_a^x f = \int_a^x t^2\, dt \quad (\text{not} \int_a^x x^2\, dx).$$

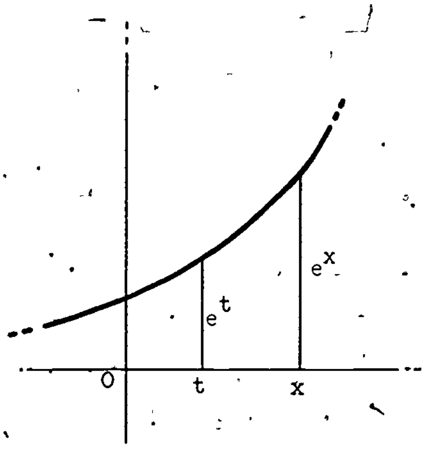
Since the result $\int_a^x t^2\, dt = \frac{t^3}{3}\Big|_a^x = \frac{x^3}{3} - \frac{a^3}{3}$ does not depend on $t$ it clearly does not matter if we replace $t$ by another letter like $u$. For this reason $t$ is called a dummy variable.

In this notation, we can rephrase property (2) as follows:

(2). If $f(t) \le g(t)$ for $a \le t \le x$ then $\int_a^x f(t)dt \le \int_a^x g(t)dt$.

Let us apply this result to the graph of the exponential function. For $t \ge 0$, $1 \le e^t$. Then by (2)

$$\int_0^x 1 \cdot dt \le \int_0^x e^t\, dt,$$




that is,

$$t\Big|_0^x \le e^t\Big|_0^x$$

and hence,

$$x - 0 \le e^x - e^0$$

$$x \le e^x - 1$$

and $\quad 1 + x \le e^x.$

This is stronger than

$$1 \le e^x.$$

If we want a still stronger result, we integrate again. But to avoid confusion we write

$$1 + t \le e^t.$$

Then $$\int_0^x (1 + t)\,dt \le \int_0^x e^t\,dt$$

$$x + \frac{x^2}{2} \le e^x - 1$$

or $$1 + x + \frac{x^2}{2} \le e^x.$$

Solutions Exercises 7-1

1. $x^3\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) < A(x) < x^3\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)$

$f : x \to x^2$

(a) $\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) < A(1) < \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)$

(i) $n = 5$ $\quad \frac{36}{150} < A(1) < \frac{66}{150}$

$.24 = \frac{6}{25} < A(1) < \frac{11}{25} = .44$

(ii) $n = 100$ $\quad \frac{19,701}{60,000} < A(1) < \frac{20,301}{60,000}$

$.328 \approx \frac{6,567}{20,000} < A(1) < \frac{6,767}{20,000} \approx .338$

(b) $2^3\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) < A(2) < 2^3\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)$

(i) $n = 5$ $\quad 8\left(\frac{6}{25}\right) < A(2) < 8\left(\frac{11}{25}\right)$

$1.92 = \frac{48}{25} < A(2) < \frac{88}{25} \approx 3.52$

(ii) $n = 100$ $\quad 8\left(\frac{6,567}{20,000}\right) < A(2) < 8\left(\frac{6,767}{20,000}\right)$

$2.63 \approx \frac{6,567}{2,500} < A(2) < \frac{6,767}{2,500} \approx 2.71$

(c) $f : x \to x^2$ and $A : x \to \frac{1}{3}x^3$

(i) If $x = \frac{1}{2}$, then $A\left(\frac{1}{2}\right) = \frac{1}{24}$

(ii) If $x = 3\sqrt{3}$, then $A(3\sqrt{3}) = 27\sqrt{3}$

2. (a) Sum of the areas of the ~~interior~~ rectangles: $[f : x \to x^3]$

$$\frac{x}{n}[f(0) + f(\frac{x}{n}) + f(\frac{2x}{n}) + \dots + f(\frac{(n-1)x}{n})]$$

$$\frac{x}{n}[0 + \frac{x^3}{n^3} + \frac{2^3x^3}{n^3} + \frac{3^3x^3}{n^3} + \dots + \frac{(n-1)^3x^3}{n^3}]$$

$$\frac{x^4}{n^4}[0 + 1^3 + 2^3 + 3^3 + \dots + (n-1)^3]$$

$$\frac{x^4}{n^4} \cdot \frac{n^4}{4}(1 - \frac{2}{n} + \frac{1}{n^2})$$

$$\frac{x^4}{4}(1 - \frac{2}{n} + \frac{1}{n^2})$$

Sum of $(n - 1)$ cubes:

$$(\frac{(n-1)n}{2})^2$$

$$\frac{(n-1)^2n^2}{4}$$

$$\frac{n^4}{4}(1 - \frac{2}{n} + \frac{1}{n^2})$$

(b) Sum of the areas of the exterior rectangles: $[f : x \to x^3]$

$$\frac{x}{n}[f(\frac{x}{n}) + f(\frac{2x}{n}) + \dots + f(\frac{nx}{n})]$$

$$\frac{x}{n}[\frac{x^3}{n^3} + \frac{2^3x^3}{n^3} + \dots + \frac{n^3x^3}{n^3}]$$

$$\frac{x^4}{n^4}[1^3 + 2^3 + 3^3 + \dots + n^3]$$

$$\frac{x^4}{n^4} \cdot \frac{n^4}{4}(1 + \frac{2}{n} + \frac{1}{n^2})$$

$$\frac{x^4}{4}(1 + \frac{2}{n} + \frac{1}{n^2})$$

Sum of $n$ cubes:

$$(\frac{n(n+1)}{2})^2$$

$$\frac{n^4}{4}(1 + \frac{2}{n} + \frac{1}{n^2})$$

$\therefore$ Summarizing part (a) and part (b), we have

$$\frac{x^4}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right) < A(x) < \frac{x^4}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

When $n \to \infty$, we have $\frac{x^4}{4} \leq A(x) \leq \frac{x^4}{4}$ ; i.e., $A : x \to \frac{1}{4}x^4$.

(c) $\frac{x^4}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right) < A(x) < \frac{x^4}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$

$\frac{1}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right) < A(1) < \frac{1}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$

(i) $n = 5$, $.16 = \frac{4}{25} < A(1) < \frac{9}{25} = .36$

(ii) $n = 100$ $\frac{9,801}{40,000} < A(1) < \frac{10,201}{40,000}$

$$.245 < A(1) < .255$$

(d) $16\left[\frac{1}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right)\right] < A(2) < 16\left[\frac{1}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\right]$

$n = 100$ $\frac{9,801}{2,500} < A(2) < \frac{10,201}{2,500}$

$$3.92 < A(2) < 4.08$$

(e) $f : x \to x^3$ and $A : x \to \frac{1}{4}x^4$

(i) If $x = 0.4$, then $A(0.4) = 0.0064$

(ii) If $x = 5\sqrt{2}$, then $A(5\sqrt{2}) = 625$

3. The area under the curve $y = 1$: $A(1) = 1$

The area under the curve $y = x^3$: $A(1) = \frac{1}{4}$

Therefore, the area of the shaded region is $1 - \frac{1}{4} = \frac{3}{4}$.

4. The intersection points are $(0,0)$ and $(1,0)$.

The area under the curve $y = x$: $A(1) = \frac{1}{2}$

The area under the curve $y = x^2$: $A(1) = \frac{1}{3}$

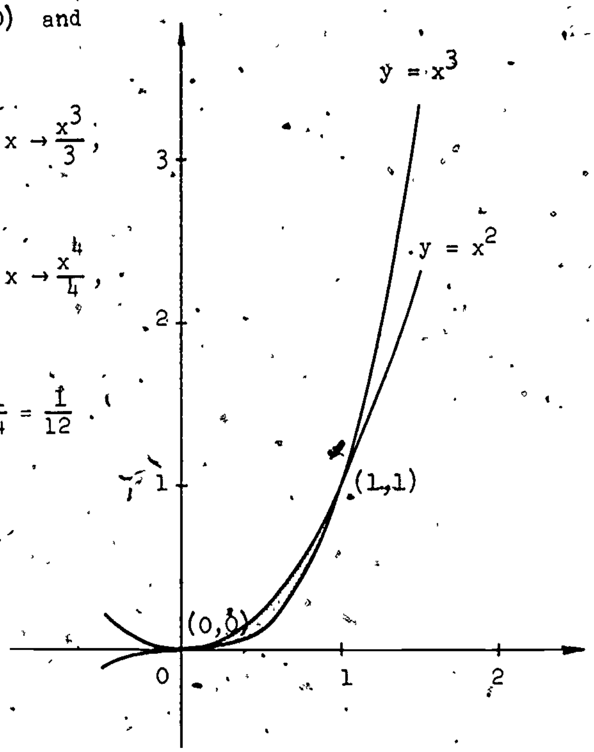
The area between the curves is $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

5. Intersection points are $(0,0)$ and $(1,1)$

$$\begin{cases} \text{If } f : x \to x^2 \text{ and } A : x \to \dfrac{x^3}{3}, \\ \text{then } A(1) = \dfrac{1}{3}. \end{cases}$$

$$\begin{cases} \text{If } g : x \to x^3 \text{ and } A : x \to \dfrac{x^4}{4}, \\ \text{then } A(1) = \dfrac{1}{4}. \end{cases}$$

Area of shaded region $= \dfrac{1}{3} - \dfrac{1}{4} = \dfrac{1}{12}$.

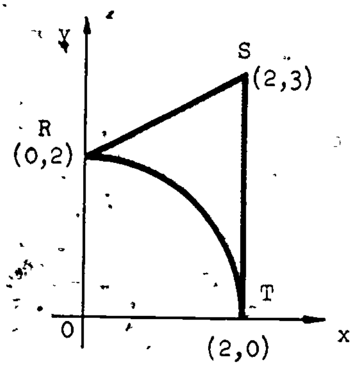



6. $f : x \to \dfrac{x}{2} + 2;\quad A : x \to \dfrac{1}{4}x^2 + 2x$

$A(2) = 1 + 4 = 5$.

$$\begin{cases} \text{Area of quarter circle} \\ \quad \dfrac{\pi(2)^2}{4} = \pi \end{cases}$$

$$\begin{cases} \text{Area of trapezoid ORST:} \\ \dfrac{1}{2}(2)(2+3) = 5 \end{cases}$$




$\therefore$ Area of shaded region: $5 - \pi \approx 1.86$ sq. units.

7. First, find area under outer parabola in quadrant 1:

$$f : x \to -x^2 + 9; \quad A : x \to -\frac{1}{3}x^3 + 9x$$

$$A(3) = -9 + 27 = 18$$

Then, find area under inner parabola in quadrant 1:

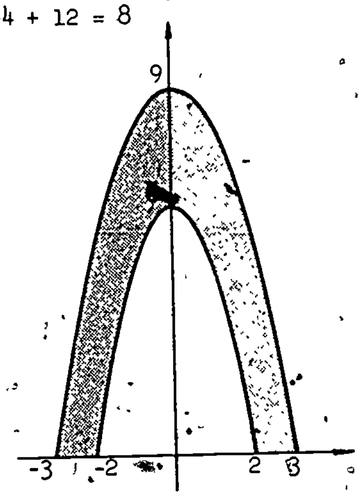
$$f : x \to -\frac{3}{2}x^2 + 6; \quad A : x \to -\frac{1}{2}x^3 + 6x$$

$$A(2) = -4 + 12 = 8$$

Therefore, by subtracting and doubling (making use of symmetry) we have:

Area of shaded region =

$2(18 - 8) = 20$ sq. units.




8. (a) We average the sum of the areas of the exterior and interior rectangles:

$$\therefore \quad A(x) \approx \frac{x^3}{3}\left(1 + \frac{1}{2n^2}\right)$$

If $n = 5$, $A(x) \approx \frac{x^3}{3}\left(1 + \frac{1}{50}\right) \approx \frac{17}{50}x^3$

(b) Adding the areas of the five trapezoids we get

$$A(x) = \frac{1}{2}\cdot\frac{x}{5}[f(0) + f(\tfrac{x}{5})] + \frac{1}{2}\cdot\frac{x}{5}[f(\tfrac{x}{5}) + f(\tfrac{2x}{5})] + \dots + \frac{1}{2}\cdot\frac{x}{5}[f(\tfrac{4x}{5}) + f(\tfrac{5x}{5})]$$

$$= \frac{x}{5}\cdot\frac{1}{2}[f(0) + 2f(\tfrac{x}{5}) + 2f(\tfrac{2x}{5}) + 2f(\tfrac{3x}{5}) + 2f(\tfrac{4x}{5}) + f(\tfrac{5x}{5})]$$

$$= \frac{x}{5}[\tfrac{1}{2}f(0) + f(\tfrac{x}{5}) + f(\tfrac{2x}{5}) + f(\tfrac{3x}{5}) + f(\tfrac{4x}{5}) + \tfrac{1}{2}f(\tfrac{5x}{5})]$$

$$= \frac{x}{5}[0 + (\tfrac{x}{5})^2 + (\tfrac{2x}{5})^2 + (\tfrac{3x}{5})^2 + (\tfrac{4x}{5})^2 + \tfrac{1}{2}(\tfrac{5x}{5})^2]$$

$$= \frac{x^3}{125}\left(0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \frac{1}{2}5^2\right) = \frac{x^3}{125}\cdot\frac{85}{2} = \frac{17x^3}{50}$$

(c) Adding rectangles with height at midpoint of intervals:

$$A(x) = \frac{x}{5}[f(\frac{x}{10}) + f(\frac{3x}{10}) + f(\frac{5x}{10}) + f(\frac{7x}{10}) + f(\frac{9x}{10})]$$

$$= \frac{x}{5}[(\frac{x}{10})^2 + (\frac{3x}{10})^2 + (\frac{5x}{10})^2 + (\frac{7x}{10})^2 + (\frac{9x}{10})^2]$$

$$= \frac{x^3}{5 \cdot 10^2}(1^2 + 3^2 + 5^2 + 7^2 + 9^2) = \frac{x^3}{5 \cdot 10^2} \cdot 165 = \frac{33x^3}{100}$$

(d) Estimates (a) and (b) are the same, a fact we might suspect from elementary geometry. By comparing the fractions $\frac{17}{50}$ and $\frac{33}{100}$ to $\frac{1}{3}$, we see that the midpoint formula is slightly better than the trapezoid formula,

i.e.,

$$\frac{33}{100} < \frac{1}{3} < \frac{17}{50}$$

$$\frac{99}{300} < \frac{100}{300} < \frac{102}{300}.$$

There is an error of $\frac{2}{300}$ in using the trapezoid (or averaging interior and exterior rectangles); there is an error of $\frac{1}{300}$ in using the midpoint formula.

Solutions Exercises 7-2

1. $A(x) \approx \frac{x^3}{3} + \frac{x^3}{6n^2}$, $n = 10$

(a) $A(2) \approx \frac{8}{3} + \frac{8}{600}$

$\approx \frac{1608}{600} \approx 2.68$

(b) $A(2.1) \approx \frac{9.261}{3} + \frac{9.261}{600}$

$\approx \frac{1852.1}{600} + \frac{9.261}{600}$

$\approx \frac{1861}{600} \approx 3.10$

(c) $\frac{A(2.1) - A(2)}{0.1} \approx \frac{\frac{1861}{600} - \frac{1608}{600}}{0.1}$

$\approx \frac{253}{60} \approx 4.2$

or by decimals

$\approx \frac{3.10 - 2.68}{0.1}$

$\approx \frac{0.42}{0.1} = 4.2$

(d) $A(x + h) = \frac{(x + h)^3}{3} + \frac{(x + h)^3}{600}$

$= (x + h)^3(\frac{200 + 1}{600})$

$= \frac{201}{600}(x + h)^3$

$A(x) = \frac{x^3}{3} + \frac{x^3}{600}$

$= \frac{201}{600}(x^3)$

$A(x + h) - A(x) = \frac{201}{600}[(x + h)^3 - x^3]$

$= \frac{201}{600}(3hx^2 + 3h^2x + h^3)$

$\frac{A(x + h) - A(x)}{h} = \frac{201}{600}(3x^2 + 3hx + h^2)$

(e) $\lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = \frac{201}{600}(3x^2)$

$$= \frac{201}{200} x^2$$

$$\approx x^2$$

We have found an approximation of the derivative, $A'(x) = x^2$.

2. $f : x \to x^2 + 1$ then $\int_0^x f = \frac{x^3}{3} + x$

(a) $\lim_{h \to 0} \int_1^{1+h} f = \lim_{h \to 0} \left[\left(\frac{(1 + h)^3}{3} + (1 + h)\right) - \left(\frac{1}{3} + 1\right)\right]$

$$= \lim_{h \to 0} \left[\left(\frac{1 + 3h + 3h^2 + h^3}{3} + 1 + h\right) - \left(\frac{1}{3} + 1\right)\right]$$

$$= \left(\frac{1}{3} + 1\right) - \left(\frac{1}{3} + 1\right)$$

$$= 0$$

The more alert student will see that we are integrating from one to one and immediately conclude the answer without calculations.

(b) $\lim_{h \to 0} \frac{1}{h} \int_1^{1+h} f = \lim_{h \to 0} \left[\frac{\frac{1 + 3h + 3h^2 + h^3}{3} + (1 + h) - \left(\frac{1}{3} + 1\right)}{h}\right]$

$$= \lim_{h \to 0} \frac{6h + 3h^2 + h^3}{3h}$$

$$= \lim_{h \to 0} 2 + h + \frac{h^2}{3}$$

$$= 2$$

The observant student will see that he has taken the derivative of the area function ~~at~~ x = 1.

$$\lim_{h \to 0} \frac{1}{h} \int_1^{1+h} f = \lim_{h \to 0} \frac{A(1 + h) - A(1)}{h}$$

$$= A'(1) = f(1) = 2.$$

(c) No. See comments after (a) and (b).

3. $F(x) = \int_{2}^{x} f$, where $f : x \to x^3$

$F(x) = \frac{x^4}{4} - \frac{2^4}{4}$

$= \frac{x^4}{4} - 4$

(a) $F(2) = 4 - 4$

$= 0$

(b) $F'(x) = 4\,\frac{x^3}{4} - 0$

$= x^3$

$F'(3) = 27$

(c) No. $F(2) = \int_{2}^{2} f = 0$. No matter what function we consider,

$\int_{a}^{a} f$ is always zero.

In part (b) we take the derivative of an antiderivative and evaluate at $x = 3$. By the Area Theorem $A'(x) = f(x)$.

4. $g' : x \to 3x^2$

$g : x \to x^3 + c$ for various values of $c$. The functions only differ by a constant.

5. (a) $f : x \to x^2$

$F(x) = \int_{0}^{x} f = \frac{x^3}{3}$

$F(2) = \frac{8}{3}$

(b) $f : x \to 2x + 1$

$F(x) = x^2 + x$

$F(2) = 6$

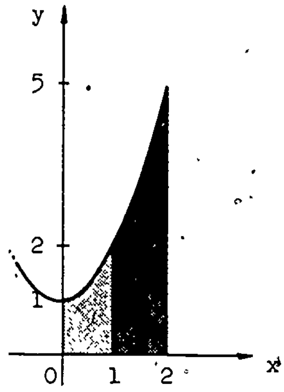
(c) $f : x \to 4x^3 + x$

$F(x) = x^4 + \frac{x^2}{2}$

$F(2) = 18$

6. $f : x \to x^2 + 1 \qquad \int_0^x f = \frac{x^3}{3} + x$

(a) See graph.



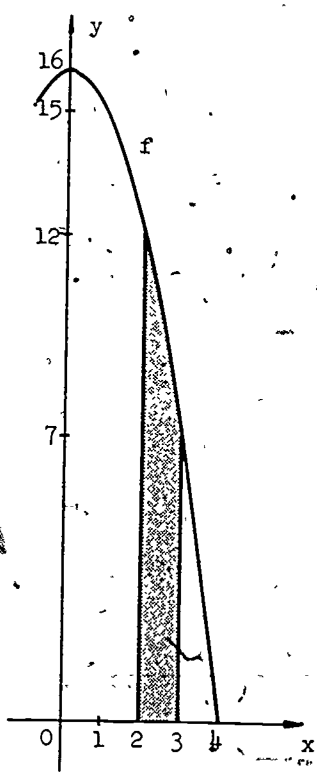

(b) $\int_0^1 f = \frac{1}{3} + 1 = \frac{4}{3}$

(c) $\int_0^2 f = \frac{8}{3} + 2 = \frac{14}{3}$

(d) The region of part (d) is equivalent to the region of part (c) with the region of part (b) removed.

$$\int_1^2 f = \frac{14}{3} - \frac{4}{3}$$
$$= \frac{10}{3}$$

7. (a) $f : x \to 16 - x^2$




$A(x) = 16x - \frac{x^3}{3}$

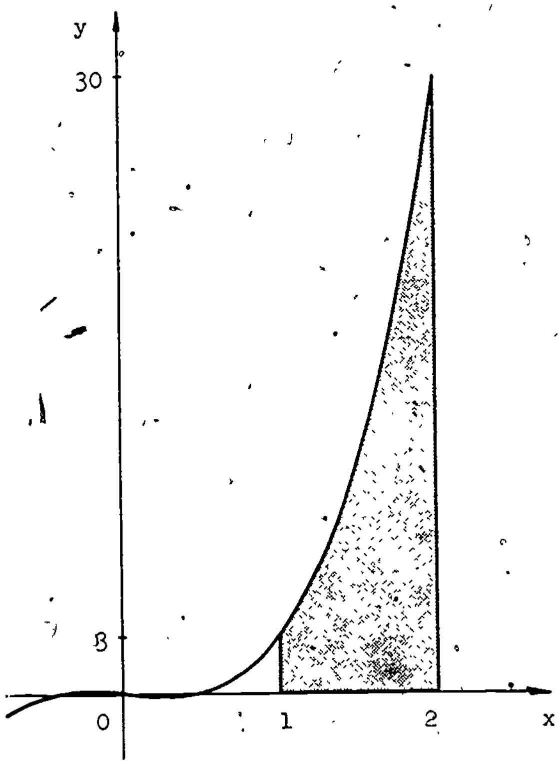
$$\int_2^3 f = (48 - 9) - (32 - \frac{8}{3})$$
$$= \frac{23}{3}$$

(b) $f : x \to 4x^3 - x$

$$A(x) = x^4 - \frac{x^2}{2}$$

$$\int_1^2 f = (16 - 1) - (1 - \frac{1}{2})$$

$$= \frac{29}{2}$$



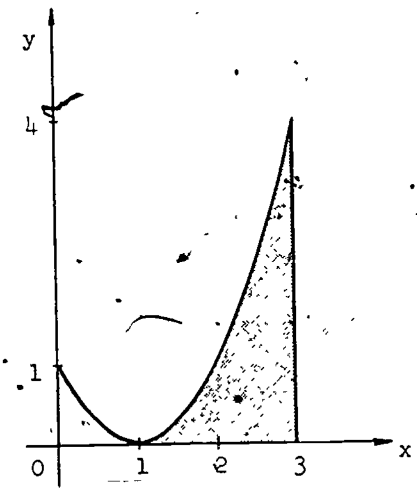

8. $f : x \to (x - 1)^2$

$$\int_0^3 f = \int_0^1 f + \int_1^3 f$$

$f$ is decreasing when $0 \leq x < 1$

and

$f$ is increasing when $1 < x \leq 3$.



## Solutions Exercises 7-3

1. (a) $$\int_0^2 (x^2 + x + 3)dx = \int_0^2 x^2\, dx + \int_0^2 x\, dx + \int_0^2 3\, dx$$
$$= \frac{x^3}{3}\Big|_0^2 + \frac{x^2}{2}\Big|_0^2 + 3x\Big|_0^2$$
$$= (\frac{8}{3} - 0) + (\frac{4}{2} - 0) + (6 - 0)$$
$$= \frac{32}{3}$$

(b) $$\int_{-2}^0 (x^2 + x + 3)dx = \int_{-2}^0 x^2\, dx + \int_{-2}^0 x\, dx + \int_{-2}^0 3\, dx$$
$$= \frac{x^3}{3}\Big|_{-2}^0 + \frac{x^2}{2}\Big|_{-2}^0 + 3x\Big|_{-2}^0$$
$$= (0 - \frac{-8}{3}) + (0 - \frac{4}{2}) + (0 - (-6))$$
$$= \frac{20}{3}$$

(c) $$\int_{-2}^2 (x^2 + x + 3)dx = \int_{-2}^2 x^2\, dx + \int_{-2}^2 x\, dx + \int_{-2}^2 3dx$$
$$= \frac{x^3}{3}\Big|_{-2}^2 + \frac{x^2}{2}\Big|_{-2}^2 + 3x\Big|_{-2}^2$$
$$= (\frac{8}{3} - \frac{-8}{3}) + (\frac{4}{2} - \frac{4}{2}) + (6 - (-6))$$
$$= \frac{52}{3}$$

(d) $$\int_0^{\pi/3} \cos x\, dx = \sin x\Big|_0^{\pi/3}$$
$$= \sin\frac{\pi}{3} - \sin 0$$
$$= (\frac{\sqrt{3}}{2}) - (0)$$
$$= \frac{\sqrt{3}}{2}$$

(e) $$\int_0^2 \sqrt{x^3}\, dx = \int_0^2 x^{3/2}\, dx$$

$$= \frac{2}{5} x^{5/2} \Big|_0^2$$

$$= \frac{2}{5}(4\sqrt{2} - 0)$$

$$= \frac{8\sqrt{2}}{5}$$

(f) $$\int_{1/16}^1 (\sqrt{x} + \sqrt[4]{x})\, dx = \int_{1/16}^1 x^{1/2}\, dx + \int_{1/16}^1 x^{1/4}\, dx$$

$$= \frac{2}{3} x^{3/2} \Big|_{1/16}^1 + \frac{4}{5} x^{5/4} \Big|_{1/16}^1$$

$$= \frac{2}{3}(1 - \frac{1}{64}) + \frac{4}{5}(1 - \frac{1}{32})$$

$$= \frac{229}{160}$$

(g) $$\int_{1/2}^1 \frac{1}{3x^2}\, dx = \int_{1/2}^1 \frac{1}{3} x^{-2}\, dx$$

$$= -\frac{1}{3} x^{-1} \Big|_{1/2}^1$$

$$= (-\frac{1}{3}) - (-\frac{1}{3} \cdot \frac{2}{1})$$

$$= \frac{1}{3}$$

(h) $$\int_{-2}^{-1} (5x^{-6} + x^2)\, dx = \int_{-2}^{-1} 5x^{-6}\, dx + \int_{-2}^{-1} x^2\, dx$$

$$= -x^{-5} \Big|_{-2}^{-1} + \frac{x^3}{3} \Big|_{-2}^{-1}$$

$$= -(\frac{1}{-1} - (\frac{1}{-32})) + ((\frac{-1}{3}) - (\frac{-8}{3}))$$

$$= \frac{317}{96}$$

(i) $\int_1^2 \frac{1}{x}\,dx = \log_e x \Big|_1^2$

$= \log_e 2 - \log_e 1$

$= \log_e 2$

(j) $\int_1^\pi x^\pi\,dx = \frac{x^{\pi+1}}{\pi+1}\Big|_1^\pi$

$= \frac{(\pi)^{\pi+1}}{\pi+1} - \frac{1}{\pi+1}$

$= \frac{\pi^{\pi+1} - 1}{\pi+1}$

(k) $\int_{-1}^2 e^x\,dx = e^x \Big|_{-1}^2$

$= e^2 - \frac{1}{e}$

$= \frac{e^3 - 1}{e}$

(ℓ) $\int_{-1}^2 (e^x + 1)dx = \int_{-1}^2 e^x\,dx + \int_{-1}^2 dx$

$= e^x \Big|_{-1}^2 + x \Big|_{-1}^2$

$= (e^2 - \frac{1}{e}) + (2 - (-1))$

$= \frac{e^3 - 1}{e} + 3$ or $\frac{e^3 + 3e - 1}{e}$

(m) $\int_{-1}^2 (e^x + x)dx = \int_{-1}^2 e^x\,dx + \int_{-1}^2 x\,dx$

$= e^x \Big|_{-1}^2 + \frac{x^2}{2}\Big|_{-1}^2$

$= (e^2 - \frac{1}{e}) + (2 - \frac{1}{2})$

$= \frac{2e^3 + 3e - 2}{2e}$

(n) $\int_1^2 (5x^4 + 3x^2 + 1)dx = \int_1^2 5x^4\,dx + \int_1^2 3x^2\,dx + \int_1^2 dx$

$= x^5 \Big|_1^2 + x^3 \Big|_1^2 + x \Big|_1^2$

$= (32 - 1) + (8 - 1) + (2 - 1)$

$= 39$

(o) $\int_{\pi/6}^{\pi/3} (\sin x + \cos x)dx = \int_{\pi/6}^{\pi/3} \sin x\,dx + \int_{\pi/6}^{\pi/3} \cos x\,dx$

$= -\cos x \Big|_{\pi/6}^{\pi/3} + \sin x \Big|_{\pi/6}^{\pi/3}$

$= ((-\frac{1}{2}) - (-\frac{\sqrt{3}}{2})) + ((\frac{\sqrt{3}}{2}) - (\frac{1}{2}))$

$= \sqrt{3} - 1$

(p) $\int_0^{4\pi/3} (e^x + \sin x)dx = \int_0^{4\pi/3} e^x\,dx + \int_0^{4\pi/3} \sin x\,dx$

$= e^x \Big|_0^{4\pi/3} + (-\cos x) \Big|_0^{4\pi/3}$

$= (e^{4\pi/3} - 1) + (-(-\frac{1}{2}) - (-1))$

$= e^{4\pi/3} + \frac{1}{2}$

(q) $\int_3^3 (x^2 + 2x + 5)dx = 0$

(r) $\int_{10}^{10} \tan x\,dx = 0$

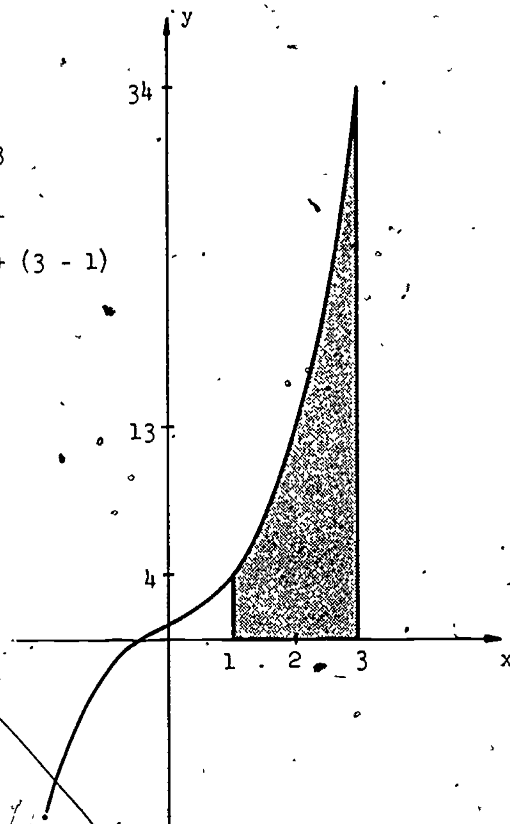
2. (a) $f : x \to x^3 + 2x + 1$

$a = 1, \quad b = 3$

$$\int_1^3 f = \frac{x^4}{4}\Big|_1^3 + x^2\Big|_1^3 + x\Big|_1^3$$

$$= \left(\frac{81}{4} - \frac{1}{4}\right) + (9 - 1) + (3 - 1)$$

$$= 30$$



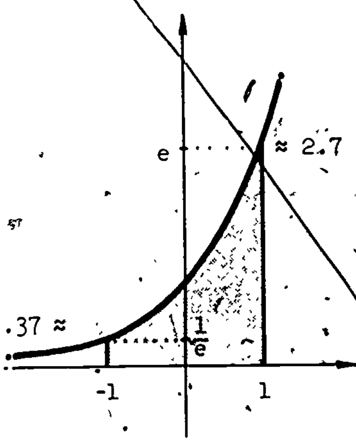

(b) $f : x \to e^x$

$a = -1, \quad b = 1$

$$\int_{-1}^1 f = e^x\Big|_{-1}^1 .$$

$$= e - \frac{1}{e}$$

$$= \frac{e^2 - 1}{e}$$



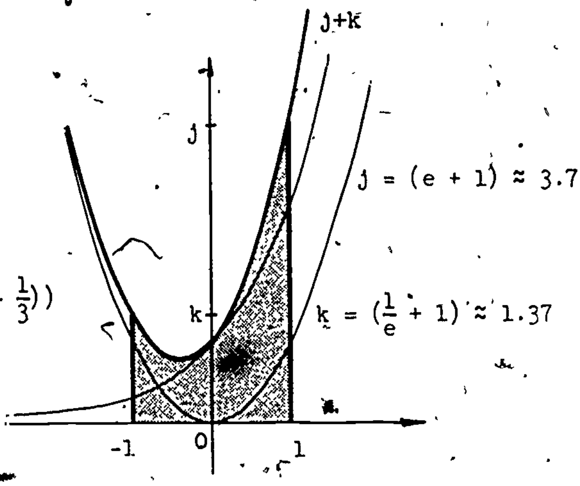
(c) $f : x \to e^x + x^2$

$a = -1, \quad b = 1$

$$\int_{-1}^{1} f = e^x \Big|_{-1}^{1} + \frac{x^3}{3}\Big|_{-1}^{1}$$

$$= (e - \frac{1}{e}) + (\frac{1}{3} - (-\frac{1}{3}))$$

$$= \frac{3e^2 + 2e - 3}{3e}$$



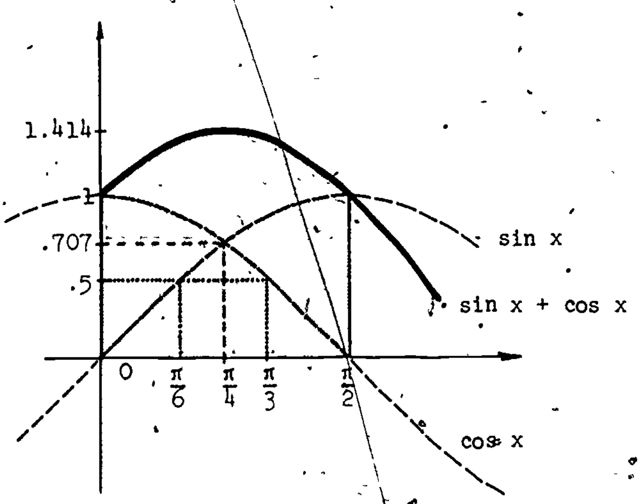

(d) $f : x \to \sin x + \cos x$

$a = 0, \quad b = \frac{\pi}{2}$

$$\int_{0}^{\pi/2} f = -\cos x \Big|_{0}^{\pi/2} + \sin x \Big|_{0}^{\pi/2}$$

$$= (-0 - (-1)) + (1 - (0))$$

$$= 2$$



(e) $f: x \to 2x^4 + \cos x$

$a = -\frac{\pi}{2}$, $b = \frac{\pi}{4}$

$$\int_{-\pi/2}^{\pi/4} f = \left(\frac{2}{5}x^5 + \sin x\right)\Bigg|_{-\pi/2}^{\pi/4}$$

$$= \left(\frac{2}{5}\left(\frac{\pi^5}{1024}\right) + \frac{\sqrt{2}}{2}\right) - \left(\frac{2}{5}\left(\frac{-\pi^5}{32}\right) - (-1)\right)$$

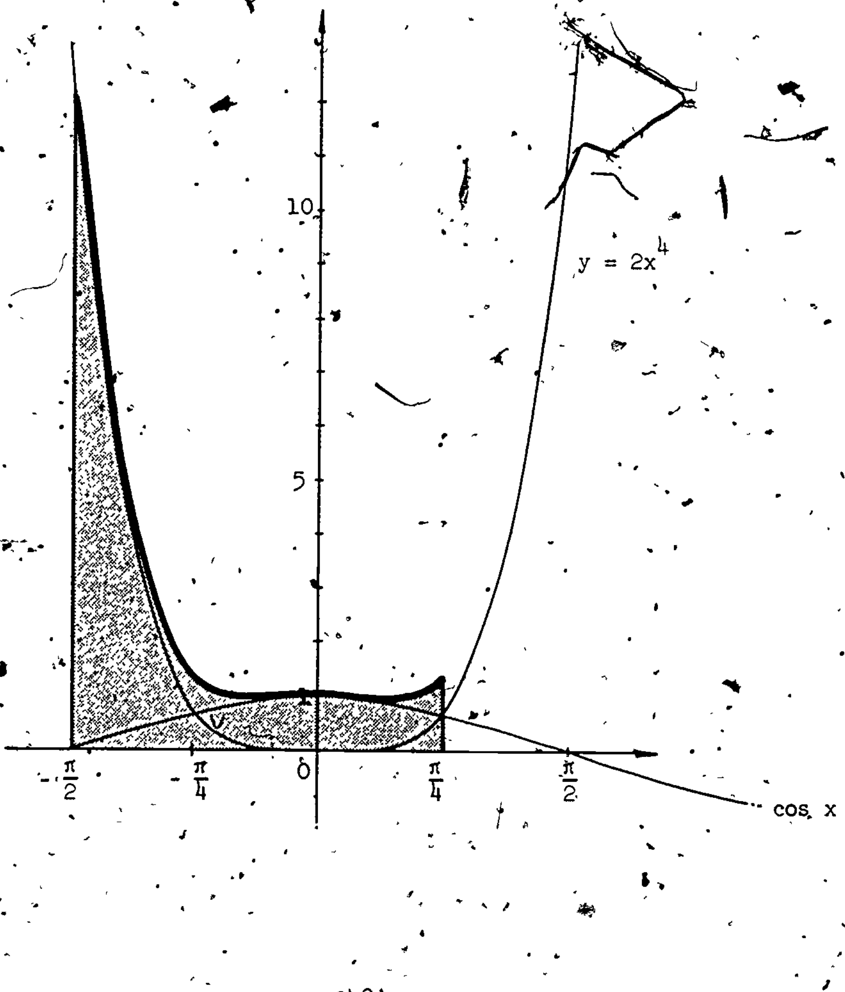
$$= \frac{2}{5}\,\frac{33}{1024}\,\pi^5 + \frac{\sqrt{2}+2}{2}$$

$$= \frac{33\pi^5}{2560} + \frac{\sqrt{2}+2}{2}$$

$$\approx \frac{33(305)}{2560} + \frac{3.414}{2}$$

$$\approx 3.9 + 1.707$$

$$\approx 5.6$$



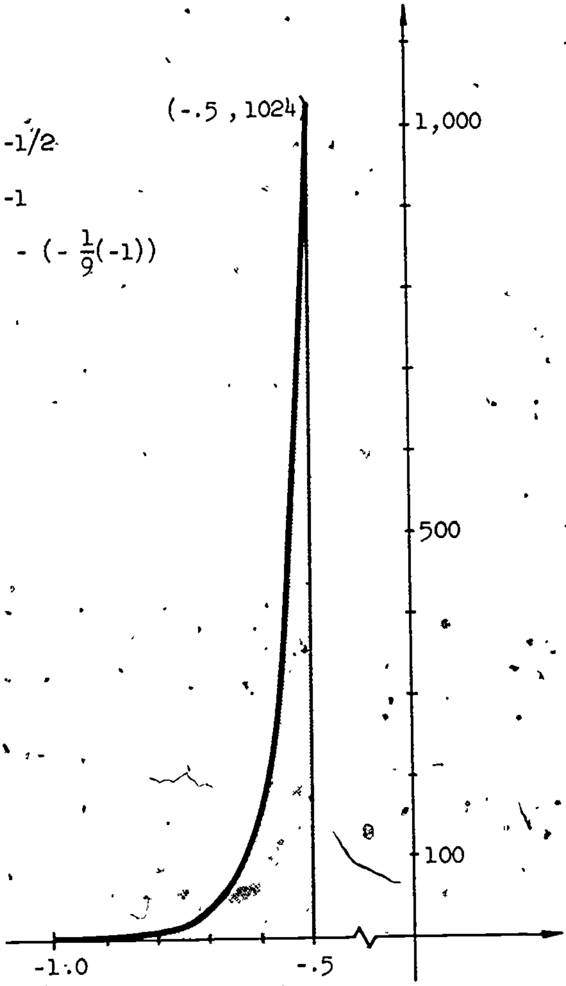
(f) $f : x \to x^{-10}$

$a = -1, \quad b = -\frac{1}{2}$

$$\int_{-1}^{-1/2} f = -\frac{1}{9}x^{-9}\Big|_{-1}^{-1/2}$$

$$= -\frac{1}{9}(-512) - (-\frac{1}{9}(-1))$$

$$= \frac{511}{9}$$



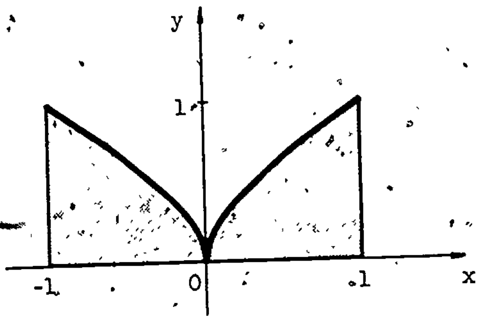

(g) $f : x \to \sqrt[3]{x^2} = x^{2/3}$

$a = -1, \quad b = 1$

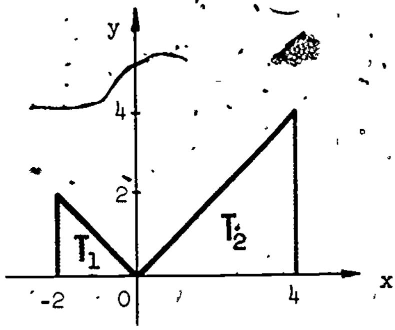
$$\int_{-1}^{1} f = \frac{3}{5}x^{5/3}\Big|_{-1}^{1}$$

$$= \frac{3}{5} - (-\frac{3}{5})$$

$$= \frac{6}{5}$$



3. (a) $f : x \to |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$




$x = -2$

$x = 4$

$$\int_{-2}^{4} f = \int_{-2}^{0} -x \, dx + \int_{0}^{4} x \, dx$$

$$= -\frac{x^2}{2}\Bigg|_{-2}^{0} + \frac{x^2}{2}\Bigg|_{0}^{4}$$

$$= (0 - (-2)) + (8 - 0)$$

$$= 10$$

By elementary geometry the area is equal to the area of two triangles $T_1$ and $T_2$.

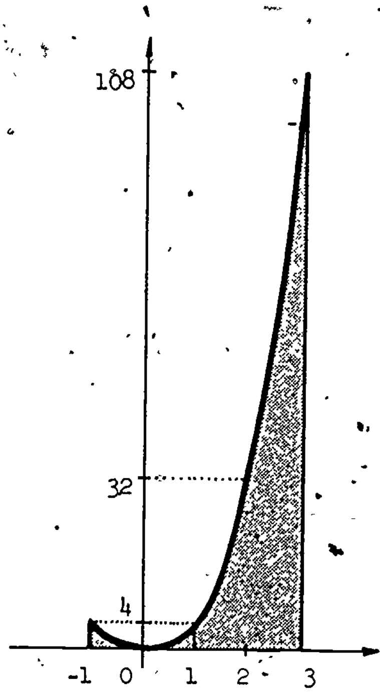
$$\text{Area} = aT_1 + aT_2$$

$$= \frac{1}{2}(2 \cdot 2) + \frac{1}{2}(4 \cdot 4)$$

$$= 2 + 8$$

$$= 10$$

(b) $f : x \to |4x^3| = \begin{cases} 4x^3, & x \geq 0 \\ -4x^3, & x < 0 \end{cases}$




$x = -1$

$x = 3$

$$\int_{-1}^{3} f = \int_{-1}^{3} -4x^3 \, dx + \int_{-1}^{3} 4x^3 \, dx$$

$$= -x^4 \Big|_{-1}^{0} + x^4 \Big|_{0}^{3}$$

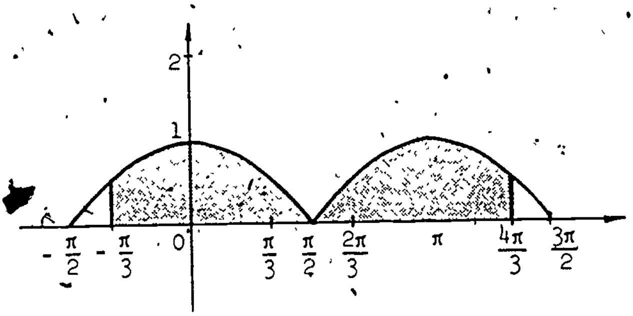
$$= (0 - (-1)) + (81 - 0)$$

$$= 82$$

(c) $f : x \to |\cos x| = \begin{cases} \cos x, & -\frac{\pi}{3} \le x \le \frac{\pi}{3} \\ -\cos x, & \frac{\pi}{2} \le x \le \frac{4\pi}{3} \end{cases}$

$x = -\frac{\pi}{3}$

$x = \frac{4\pi}{3}$




$$\int_{-\pi/3}^{4\pi/3} f = \int_{-\pi/3}^{\pi/2} \cos x \, dx + \int_{\pi/2}^{4\pi/3} - \cos x \, dx$$

$$= \sin x \Big|_{-\pi/3}^{\pi/2} + (-\sin x) \Big|_{\pi/2}^{4\pi/3}$$

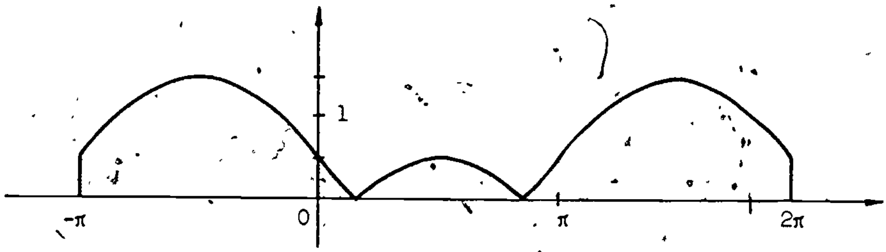
$$= \left(1 - \left(-\frac{\sqrt{3}}{2}\right)\right) + \left(\left(-\left(-\frac{\sqrt{3}}{2}\right) - (-(+1))\right)\right)$$

$$= 2 + \sqrt{3}$$

*(d) $f : x \to |\frac{1}{2} - \sin x| = \begin{cases} (\frac{1}{2} - \sin x), & -\pi \le x < \frac{\pi}{6} \text{ and } \frac{5\pi}{6} \le x \le 2\pi \\ (-\frac{1}{2} + \sin x), & \pi \le x \le \frac{\pi}{6} \end{cases}$

$x = -\pi$

$x = 2\pi$



$$\int_{-\pi}^{2\pi} f = \int_{-\pi}^{\pi/6}\left(\frac{1}{2} - \sin x\right)dx + \int_{\pi/6}^{5\pi/6}\left(-\frac{1}{2} + \sin x\right)dx + \int_{5\pi/6}^{2\pi}\left(\frac{1}{2} - \sin x\right)dx$$

$$= \left(\frac{x}{2} + \cos x\right)\Big|_{-\pi}^{\pi/6} + \left(-\frac{x}{2} - \cos x\right)\Big|_{\pi/6}^{5\pi/6} + \left(\frac{x}{2} + \cos x\right)\Big|_{5\pi/6}^{2\pi}$$

$$= \left[\left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right) - \left(-\frac{\pi}{2} + (-1)\right)\right] + \left[\left(\frac{-5\pi}{12} - \left(-\frac{\sqrt{3}}{2}\right)\right) - \left(\frac{-\pi}{12} - \frac{\sqrt{3}}{2}\right)\right]$$
$$+ \left[(\pi + 1) - \left(\frac{5\pi}{12} + \left(\frac{-\sqrt{3}}{2}\right)\right)\right]$$

$$= \frac{5\pi}{6} + 2 + 2\sqrt{3}$$

(e) $f : x \to |1 - \sqrt{x}| = \begin{cases} 1 - \sqrt{x}, & 0 \le x \le 1 \\ -1 + \sqrt{x}, & 1 < x \end{cases}$

$x = 0$

$x = 4$

$$\int_0^4 f = \int_0^1 (1 - \sqrt{x})dx + \int_1^4 (-1 + \sqrt{x})dx$$

$$= \left(x - \frac{2}{3}x^{3/2}\right)\Big|_0^1 + \left(-x + \frac{2}{3}x^{3/2}\right)\Big|_1^4$$

$$= \left[\left(1 - \frac{2}{3}\right) - 0\right] + \left[\left(-4 + \frac{16}{3}\right) - \left(-1 + \frac{2}{3}\right)\right]$$

$$= 2$$

4\. (a) $(x^2 + 3\sqrt{x})\Big|_1^4 = (16 + 6) - (1 + 3)$

$= 18$

$(x^2 + 3\sqrt{x} + 50)\Big|_1^4 = (16 + 6 + 50) - (1 + 3 + 50)$

$= 18$

(b) Since $F$ and $G$ differ only by a constant,

$$F(x)\Big|_0^1 = (-1) - (+1) = -2 = G(x)\Big|_0^1.$$

(c) If $F' = G'$ then $F(x)\Big|_a^b - G(x)\Big|_a^b = 0.$

5. (a) (i) $f : x \to (x - 1)^3$

$$\int f = \frac{1}{4}(x - 1)^4$$

(ii) $F : x \to x^3 - 3x^2 + 3x - 1$

$$\int F = \frac{1}{4}x^4 - x^3 + \frac{3}{2}x^2 - x$$

(iii) $g : x \to 8x^3 - 12x^2 + 6x - 1$

$$\int g = 2x^4 - 4x^3 + 3x^2 - x$$

(iv) $G : x \to (2x - 1)^3 = 8(x - \frac{1}{2})^3$

$$\int G = 8 \cdot \frac{1}{4}(x - \frac{1}{2})^4$$
$$= 2(x - \frac{1}{2})^4$$

(b) Since $f = F$ and $g = G$, we would expect their respective anti-derivatives to differ by at most a constant.

Expanding, $\int f = \frac{1}{4}(x^4 - 4x^3 + 6x^2 - 4x + 1)$
$$= \frac{1}{4}x^4 - x^3 + \frac{3}{2}x^2 - x + \frac{1}{4}.$$

We note that $\int F$ and $\int f$ differ by $\frac{1}{4}$.

Expanding, $\int G = 2(x^4 - \frac{4}{2}x^3 + \frac{6}{4}x^2 - \frac{4}{8}x + \frac{1}{16})$
$$= 2x^4 - 4x^3 + 3x^2 - x + \frac{1}{8}.$$

We see that $\int g(x)$ and $\int G(x)$ differ by $\frac{1}{8}$.

6. $f : x \to 8(x + 1)^3$

$$\int f = 8 \cdot \frac{1}{4}(x + 1)^4$$
$$= 2(x + 1)^4$$

Since $g : x \to (2x + 2)^3 = 8(x + 1)^3$

$\int g = 2(x + 1)^4$ also.

7. Find $\int_0^1 (3x + 4)^5\,dx$

(a) By expanding we have

$$F(x) = \int (243x^5 + 1620x^4 + 4320x^3 + 5760x^2 + 3840x + 1024)dx.$$

This is obviously a messy process which breeds arithmetic errors.

$$F(x) = \frac{243}{6}x^6 + \frac{1620}{5}x^5 + \frac{4320}{4}x^4 + \frac{5760}{3}x^3 + \frac{3840}{2}x^2 + 1024x$$

$$F(1) - F(0) = \frac{243}{6} + \frac{1620}{5} + \frac{4320}{4} + \frac{5760}{3} + \frac{3840}{2} + 1024$$

$$= \frac{243}{6} + 324 + 1080 + 1920 + 1920 + 1024$$

$$= 6308\frac{1}{2}$$

(b) This method should be a welcome relief after (a).

Let $(3x + 4)^5 = 243(x + \frac{4}{3})^5$

Then $F(x) = \int 243(x + \frac{4}{3})^5\,dx$

$$= 243 \cdot \frac{1}{6}(x + \frac{4}{3})^6$$

$$F(1) - F(0) = \frac{243}{6}(\frac{7}{3})^6 - \frac{243}{6}(\frac{4}{3})^6$$

$$= \frac{243}{6 \cdot 3^6}(7^6 - 4^6)$$

$$= \frac{1}{18}(117649 - 4096)$$

$$= \frac{1}{18}(113553)$$

$$= 6308\frac{1}{2}$$

8. (a), (c), and (d).

9. (a) $\int_{-\pi/6}^{\pi/6} \cos x\,dx = 2\int_0^{\pi/6} \cos x\,dx$

$$= 2\sin x\Big|_0^{\pi/6}$$

$$= 2(\frac{1}{2} - 0)$$

$$= 1$$

(b) $$\int_{-2}^{2}(1 + 6x^2)\,dx = 2\int_{0}^{2}(1 + 6x^2)\,dx$$

$$= 2(x + 2x^3)\Big|_0^2$$

$$= 2((2 + 16) - 0)$$

$$= 36$$

(c) $$\int_{0}^{2}(x - 1)^2 dx = 2\int_{0}^{1}(x - 1)^2\,dx$$

$$= \frac{2}{3}(x - 1)^3\Big|_0^1$$

$$= \frac{2}{3}(0 - (-1))$$

$$= \frac{2}{3}$$

An alternate method involves changing the function so that it is symmetric with respect to the y-axis.

$$\int_{0}^{2}(x - 1)^2\,dx = \int_{-1}^{1} x^2\,dx$$

$$= 2\int_{0}^{1} x^2\,dx$$

$$= 2\,\frac{x^3}{3}\Big|_0^1$$

$$= \frac{2}{3}(1 - 0)$$

$$= \frac{2}{3}$$

(d) $$\int_{0}^{\pi}\sin x\,dx = 2\int_{0}^{\pi/2}\sin x\,dx$$

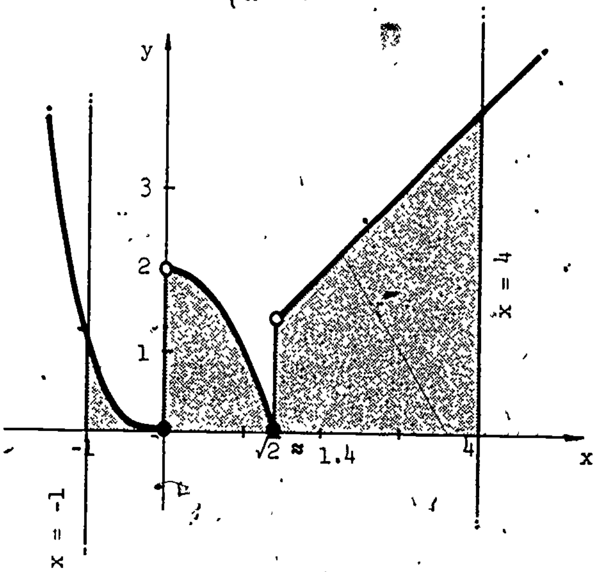
$$= -2\cos x\Big|_0^{\pi/2}$$

$$= 2(0 - (-1))$$

$$= 2$$

10. (a) $f : x \to \begin{cases} -x^3, & x \leq 0 \\ -x^2 + 2, & 0 < x \leq \sqrt{2} \\ x, & \sqrt{2} < x \end{cases}$

vertical lines $\begin{cases} x = -1 \\ x = 4 \end{cases}$




$$\int_{-1}^{4} f(x)\,dx = \int_{-1}^{0} -x^3\,dx + \int_{0}^{\sqrt{2}} (-x^2 + 2)\,dx + \int_{\sqrt{2}}^{4} x\,dx$$

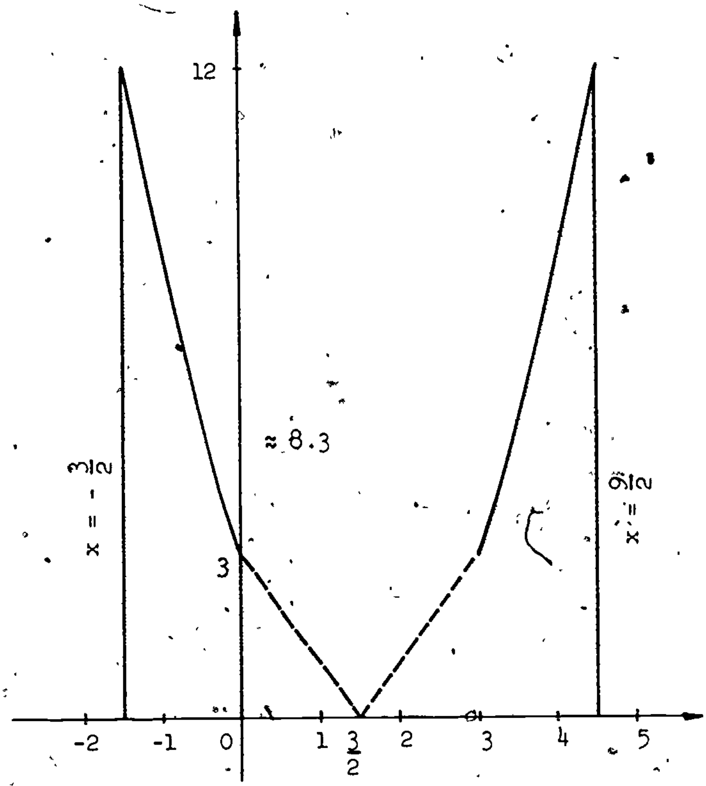
$$= -\frac{x^4}{4}\Big|_{-1}^{0} + \left(-\frac{x^3}{3} + 2x\right)\Big|_{0}^{\sqrt{2}} + \frac{x^2}{2}\Big|_{\sqrt{2}}^{4}$$

$$= \left(0 - \left(-\frac{1}{4}\right)\right) + \left(\left(-\frac{2\sqrt{2}}{3} + 2\sqrt{2}\right) - 0\right) + \left(\frac{16}{2} - \frac{2}{2}\right)$$

$$= \frac{29}{4} + \frac{4}{3}\sqrt{2}$$

(b) $f : x \to \begin{cases} |2x - 3| & \text{if } 0 \le x \le 3 \\ \frac{4}{3}(x - \frac{3}{2})^2 & \text{if } x \le 0 \text{ or } 3 \le x \end{cases}$

vertical lines $\begin{cases} x = -\frac{3}{2} \\ x = \frac{9}{2} \end{cases}$




By symmetry

$$\int_{-3/2}^{9/2} f(x)dx = 2\int_{3/2}^{9/2} f(x)dx$$

$$= 2\int_{3/2}^{3} |2x - 3|dx + 2\int_{3}^{9/2} \frac{4}{3}(x - \frac{3}{2})^2 dx$$

Since $2x - 3 \geq 0$ when $x \geq \frac{2}{3}$ then

$$|2x - 3| = 2x - 3$$

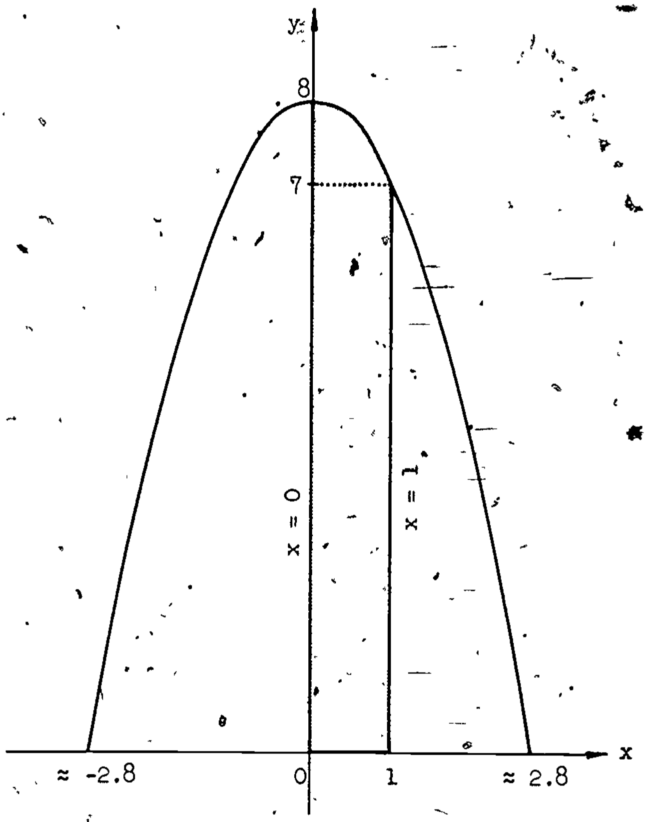
$$\int_{-3/2}^{9/2} f(x)dx = 2(x^2 - 3x)\Big|_{3/2}^{3} + \frac{8}{3}\left(\frac{1}{3}\right)\left(x - \frac{3}{2}\right)^3\Big|_{3}^{9/2}$$

$$= 2\left((9 - 9) - \left(\frac{9}{4} - \frac{9}{2}\right)\right) + \frac{8}{9}\left(\left(\frac{6}{2}\right)^3 - \left(\frac{3}{2}\right)^3\right)$$

$$= \frac{51}{2}$$

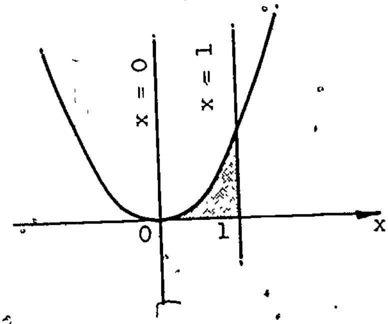
11. (a) (i) $\int_0^1 (8 - x^2)dx = \left(8x - \frac{x^3}{3}\right)\Big|_0^1$

$$= \frac{23}{3}$$



(ii) $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1$

$= \frac{1}{3}$

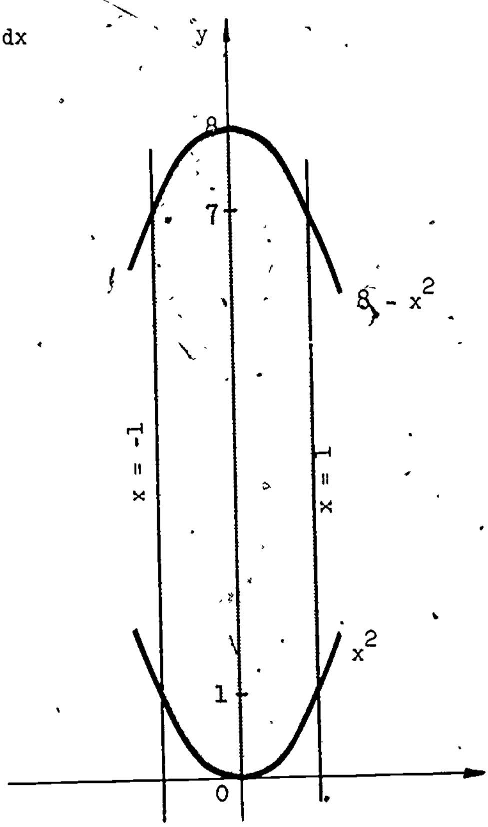



(b) The area desired is

$$2\int_0^1 (8 - x^2)dx - 2\int_0^1 x^2\,dx$$

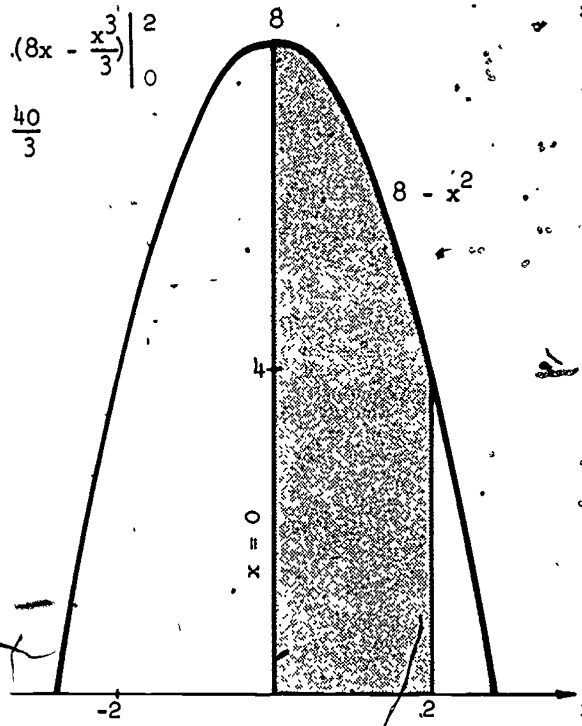
which is

$$2(\frac{23}{3}) - 2(\frac{1}{3}) = \frac{44}{3}$$



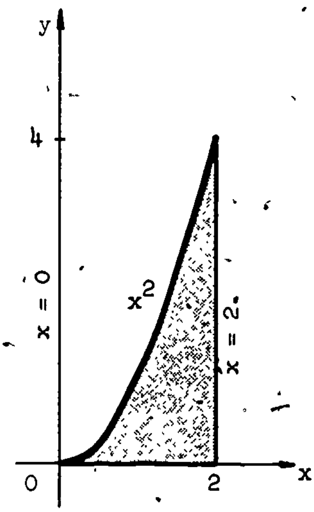
12. (a) (i) $\int_0^2 (8 - x^2)dx = (8x - \frac{x^3}{3})\Big|_0^2$

$= \frac{40}{3}$




(ii) $\int_0^2 x^2\, dx = \frac{x^3}{3}\Big|_0^2$

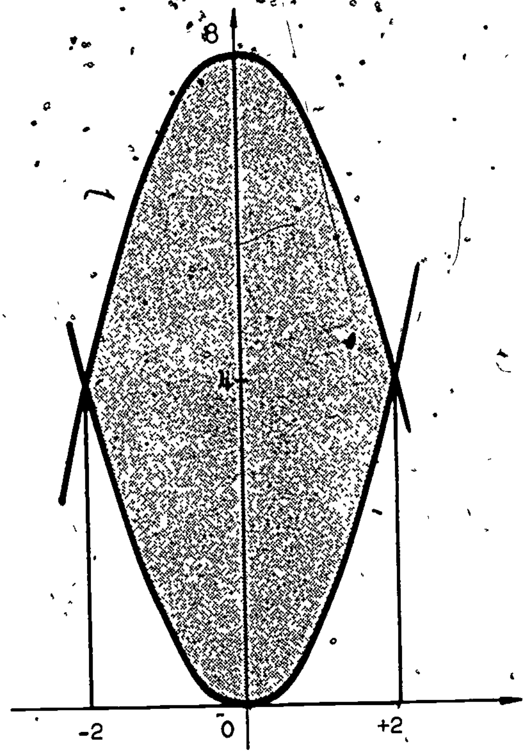
$= \frac{8}{3}$



(b) The desired area is

$$2\int (8 - x^2)dx - 2\int_0^2 x^2\,dx$$

$$2\left(\frac{40}{3}\right) - 2\left(\frac{8}{3}\right) = \frac{64}{3}$$




13. (a) $\displaystyle\int_{-1}^{1}(8 - x^2)dx - \int_{-1}^{1} x^2\,dx = \int_{-1}^{1}(8 - x^2 - x^2)dx$

$$= 2\int_0^1 (8 - 2x^2)dx$$

$$= 2\left(8x - \frac{2}{3}x^3\right)\Big|_0^1$$

$$= \frac{44}{3}$$

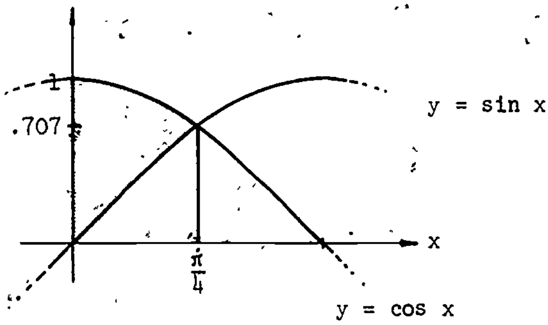
(b) $\displaystyle\int_{-2}^{2}(8 - x^2)dx - \int_{-2}^{2} x^2\,dx = \int_{-2}^{2}(8 - x^2 - x^2)dx$

$$= 2\int_0^2 (8 - 2x^2)dx$$

$$= 2\left(8x - \frac{2}{3}x^3\right)\Big|_0^2$$

$$= \frac{64}{3}$$

14. $y = \sin x, \quad y = \cos x, \quad x = 0, \quad x = \frac{\pi}{4}$




$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)dx$$

$$= (\sin x + \cos x)\Big|_0^{\pi/4}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1)$$

$$= \sqrt{2} - 1$$

## Solutions-Exercises 7-4

1. Let $F' = f$. By the Fundamental Theorem

$$\int_a^a f = F(x)\Big|_a^a = F(a) - F(a) = 0.$$

2. If $f(x) \leq g(x)$, for $a \leq x \leq b$, then

$$(g - f)(x) = g(x) - f(x) \geq 0, \quad \text{for} \quad a \leq x \leq b.$$

Property (1) then shows that

$$\int_a^b (g - f) \geq 0.$$

We may prove that this result is equivalent to

$$\int_a^b f \leq \int_a^b g$$

as follows:

$$g = f + (g - f).$$

Hence, by property (7)

$$\int_a^b g = \int_a^b f + \int_a^b (g - f)$$

and

$$\int_a^b (g - f) = \int_a^b g - \int_a^b f.$$

Hence, if

$$\int_a^b (g - f) \geq 0,$$

$$\int_a^b g - \int_a^b f \geq 0$$

and

$$\int_a^b f \leq \int_a^b g.$$

3. $\int_a^x f = F(x) - F(a)$

$$D\int_a^x f = DF(x) - DF(a)$$
$$= f(x) - 0$$
$$= f(x).$$

4. $D\int_a^x f + g = (f + g)(x) = f(x) + g(x)$.

$$D\left[\int_a^x f + \int_a^x g\right] = D\int_a^x f + D\int_a^x g$$

$$= f(x) + g(x).$$

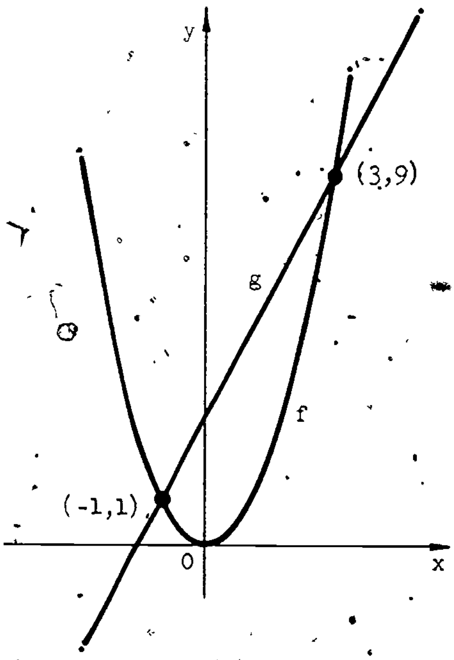
Hence,

$$\int_a^x (f+g) = \int_a^x f + \int_a^x g + C$$

where $C$ is some constant. $C = 0$ since $\int_a^a (f+g) = \int_a^a f = \int_a^a g = 0$.

5. $f : x \to x^2, \quad g : x \to 2x + 3$.

(a)




(b) The graphs of $f$ and $g$ have two points of intersection, namely $(-1,1)$ and $(3,9)$. This implies that the graph of $f$ is entirely above or entirely below the graph of $g$ when $-1 < x < 3$. We pick an interior point on the interval, $x = 0$. Evaluating $f$ and $g$ yields $f(0) = 0 < g(0) = 3$. Thus the graph of $f$ is entirely below the graph of $g$ when $-1 < x < 3$ and $f(x) \le g(x)$ when $0 \le x \le 3$.

(c) $\int_0^3 f = \int_0^3 x^2\,dx = \frac{x^3}{3}\Big|_0^3 = 9$

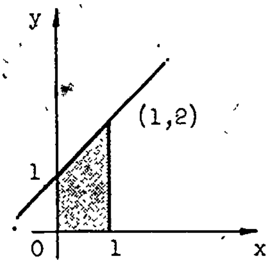
$\int_0^3 g = \int_0^3 (2x + 3)dx = x^2 + 3x\Big|_0^3 = 18$

$\therefore \int_0^3 f \leq \int_0^3 g.$

6. (a) $f : x \to x + 1$

$f' : x \to 1$

We need test only the end-points since $f'(x) \neq 0$ for all values of $x$.




$f(0) = 1 = m$

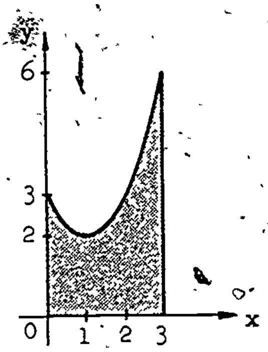
$f(1) = 2 = M$

$$m(a - 0) \leq A(1) \leq M(a - 0)$$
$$1(1 - 0) \leq A(1) \leq 2(1 - 0)$$
$$1 \leq A(1) \leq 2$$

(b) $f : x \to x^2 - 2x + 3$

$f' : x \to 2x - 2$

We must test for relative extremum, as well as end-points.




$f'(x) = 0$ when $x = 1$

$f(0) = 3,\ f(1) = 2$ and $f(3) = 6.$

Thus $m = 2$ and $M = 6$

$$2(3 - 0) \leq A(3) \leq 6(3 - 0)$$
$$6 \leq A(3) \leq 18$$

7. $\int_5^{10} f = \int_5^{10} (3x - 2)dx = \frac{3}{2}x^2 - 2x\Big|_5^{10} = \frac{205}{2}$

$\int_5^{10} g = \int_5^{10} \sqrt{2}(3x - 2)dx = \frac{3\sqrt{2}}{2}x^2 - 2\sqrt{2}\,x\Big|_5^{10} = \frac{205}{2}\sqrt{2}.$

$\therefore \int_5^{10} g = \sqrt{2}\int_5^{10} f.$

8. $f : x \to -2x + 20$ and $g : x \to -2(x - h) + 20$

(a) Select $h = -3$ then $g(x) = f(x + 3)$. Substituting we find $f(3) = g(0)$. $f(7) = 6 = g(4)$.

(b) $$\int_0^3 f = \int_0^3 (-2x + 20)dx = -x^2 + 20x \Big|_0^3 = 51$$

$$\int_0^4 g = \int_0^4 (-2(x + 3) + 20)dx = -(x + 3)^2 + 20x \Big|_0^4 = 31 + 9 = 40$$

$$\int_0^7 f = \int_0^7 (-2x + 20)dx = -x^2 + 20x \Big|_0^7 = 91$$

$$\therefore \int_0^7 f = \int_0^3 f + \int_0^4 g.$$

9. $f : x \to 3x + 5$, $g : x \to x$ and $h : x \to 1$

$$\int_a^b f = \int_a^b (3x + 5)dx = \frac{3}{2}x^2 + 5x \Big|_a^b = \frac{3}{2}b^2 + 5b - \frac{3}{2}a^2 - 5a$$

$$\int_a^b g = \int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2}{2} - \frac{a^2}{2}$$

$$\int_a^b h = \int_a^b 1\,dx = x \Big|_a^b = b - a$$

$$\therefore 3\int_a^b g + 5\int_a^b h = 3\left(\frac{b^2}{2} - \frac{a^2}{2}\right) + 5(b - a)$$

$$= \frac{3}{2}b^2 + 5b - \frac{3}{2}a^2 - 5a$$

$$= \int_a^b f.$$

10. (a) $\int_1^3 (x^2 + x)dx = \frac{x^3}{3} + \frac{x^2}{2} \Big|_1^3 = (9 + \frac{9}{2}) - (\frac{1}{3} + \frac{1}{2}) = \frac{38}{3}$

(b) $\int_1^4 (x^2 - 4x + 5)dx = \frac{x^3}{3} - 2x^2 + 5x \Big|_1^4 = (\frac{64}{3} - 32 + 20) - (\frac{1}{3} - 2 + 5) = 6$

(c) $\int_1^3 (-x^2 + 2x + 3)dx = -\frac{x^3}{3} + x^2 + 3x \Big|_1^3 = (-9 + 9 + 9) - (-\frac{1}{3} + 1 + 3) = \frac{16}{3}$

(d) $\int_2^4 (\frac{1}{4}x^2 + \frac{1}{2}x - 1)dx = \frac{1}{12}x^3 + \frac{1}{4}x^2 - x \Big|_2^4 = (\frac{64}{12} + 4 - 4) - (\frac{8}{12} + 1 - 2) = \frac{17}{3}$

11. $f : x \to px^2 + qx + r$

(a) $F : x \to \frac{p}{3}x^3 + \frac{q}{2}x^2 + rx$

$F'(x) = 3(\frac{p}{3})x^2 + 2(\frac{q}{2})x + r$

$= px^2 + qx + r$

$= f(x)$

(b) $\int_a^b f = \int_0^b f - \int_0^a f$

Since $F'(x) = f(x)$,

then $\int_0^x f = F(x)$ and $\int_a^b f = F(b) - F(a)$.

12. $g : x \to px^3 + qx^2 + rx + s$

$G : x \to \frac{p}{4}x^4 + \frac{q}{3}x^3 + \frac{r}{2}x^2 + sx$

(a) $G'(x) = 4(\frac{p}{4})x^3 + 3(\frac{q}{3})x^2 + 2(\frac{r}{2})x + s$

$= px^3 + qx^2 + rx + s$

$= g(x)$

(b) $\int_a^b g = \int_0^b g - \int_0^a g$

Since $G'(x) = g(x)$ then $\int_0^x g = G(x)$ thus $\int_a^b g = G(b) - G(a)$.

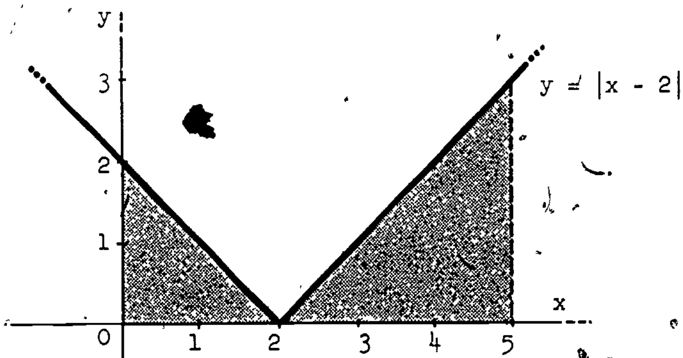
13. $G(x) = F(x) + 100$

$G'(x) = F'(x)$ implies that $G'(x) = f(x)$ also.

Since $\int_a^b f = F(b) - F(a)$ then $\int_a^b f = G(b) - G(a)$.

14. $f : x \to |x - 2|$, so

$$f : x \to \begin{cases} x - 2 & \text{if } x \geq 2 \\ -x + 2 & \text{if } x < 2 \end{cases}$$




$$\therefore \int_0^5 |x - 2|\,dx = \int_0^2 (-x + 2)\,dx + \int_2^5 (x - 2)\,dx$$

$$= -\frac{x^2}{2} + 2x \Big|_0^2 + \frac{x^2}{2} - 2x \Big|_2^5$$

$$= (-2 + 4) + (\frac{25}{2} - 10) - (2 - 4)$$

$$= \frac{13}{2}.$$

15. $\int_{-10}^{-3} x^2\,dx = \frac{x^3}{3} \Big|_{-10}^{-3} = (-\frac{27}{3}) - (-\frac{1000}{3}) = \frac{973}{3}$

or, let $g(x) = f(x + 10)$.

Then $\int_{-10}^{-3} f = \int_0^7 g$

where $g : x \to (x - 10)^2$,

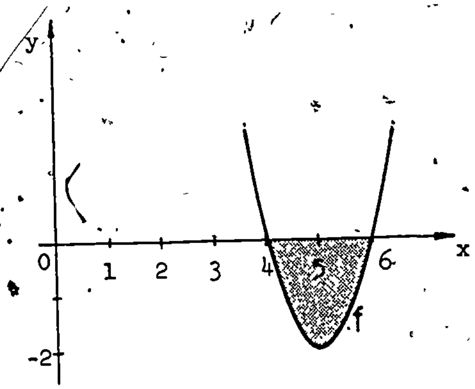
$$\int_0^7 (x - 10)^2\,dx = \frac{(x - 10)^3}{3} \Big|_0^7 = \frac{(-3)^3}{3} - \frac{(-10)^3}{3} = \frac{973}{3}.$$

16. $f(x) = y = 2(x - 5)^2 - 2$ and $y = 0$.

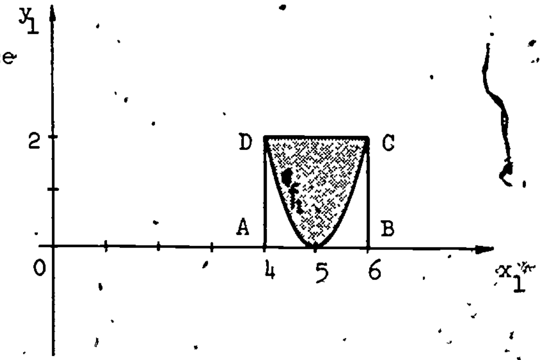
Let $y_1 = y + 2$ in order to raise the graph two units. Then

$f_1(x) = y_1 = (2(x - 5)^2 - 2) + 2$

and $f_1(x) = y_1 = 2(x - 5)^2$.




The area in question now becomes the area of $\square$ABCD less the area under the graph of y.




The area of $\square$ABCD is 4, since

$$\int_4^6 2dx = 2x \Big|_4^6 = 12 - 8 = 4.$$

The area under $f_1$ from 4 to 6 is $2\int_4^5 f_1$ by symmetry.

$$\int_4^5 f_1 = \frac{2}{3}x^3 - 10x^2 + 50x \Big|_4^5 = (\frac{2}{3} \cdot 125 - 10 \cdot 5) - (\frac{2}{3} \cdot 64 - 10 \cdot 16 + 200)$$

$$= \frac{2}{3}$$

The desired area is then the area of the rectangle, 4, less

$$2\int_4^5 f_1 = \frac{4}{3},$$

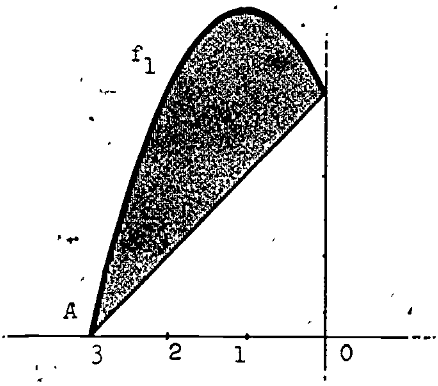
$$4 - \frac{4}{3} = \frac{8}{3}.$$

17. $f(x) = y = -(x + 1)^2 + 1$ and $g(x) = y = x$.

Raise the two graphs to obtain the figure shown.

Let $f_1(x) = -(x + 1)^2 + 1 + 3$

$$= 4 - (x + 1)^2.$$




Then

$$\int_{-3}^{0} f_1 = 4x - \frac{(x + 1)^3}{3} \Bigg|_{-3}^{0}$$

$$= (-\frac{1}{3}) - (-12 - \frac{(-8)}{3}$$

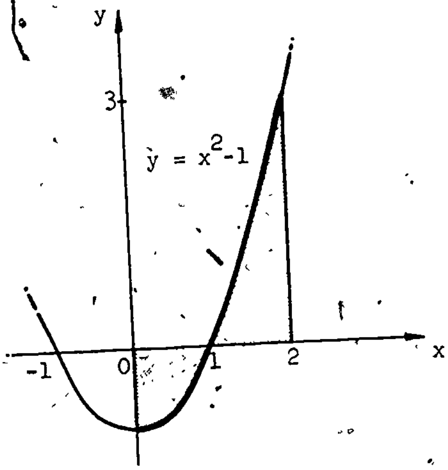
$$= -\frac{1}{3} + 12 - \frac{8}{3} = 9.$$

From this area we must subtract the area of the triangle AOB. Hence, the required area is

$$9 - \frac{9}{2} = \frac{9}{2}.$$

## Solutions Exercises 7-5

1\. (a)




(b) $\displaystyle\int_0^2 (x^2 - 1)dx = \left(\frac{x^3}{3} - x\right)\Big|_0^2 = \frac{8}{3} - 2 = \frac{2}{3}$

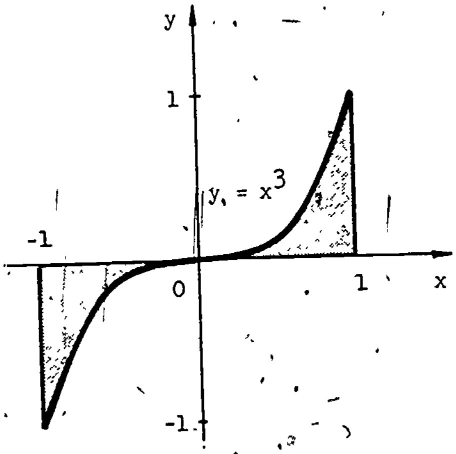
(c) $\displaystyle A = \int_0^1 -(x^2 - 1)dx + \int_1^2 (x^2 - 1)dx$

$\displaystyle = \left(-\frac{x^3}{3} + x\right)\Big|_0^1 + \left(\frac{x^3}{3} - x\right)\Big|_1^2$

$\displaystyle = \left(-\frac{1}{3} + 1\right) + \left(\frac{8}{3} - 2 - \frac{1}{3} + 1\right)$

$\displaystyle = \frac{2}{3} + \frac{4}{3} = 2$

2\. (a)



(b) $\int_{-1}^{1} x^3\,dx = \frac{1}{4}x^4 \Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$

(c) $A = \int_{-1}^{0} (-x^3)dx + \int_{0}^{1} x^3\,dx$

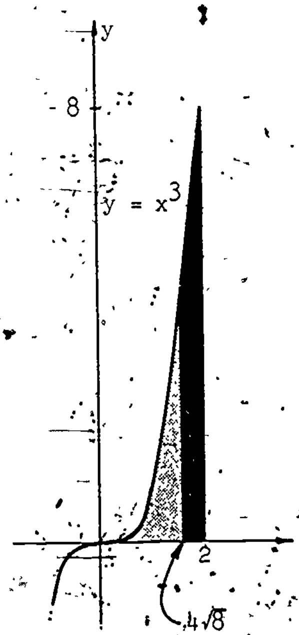
$= -\frac{x^4}{4}\Big|_{-1}^{0} + \frac{x^4}{4}\Big|_{0}^{1}$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

(d) $A = \int_{0}^{2} x^3\,dx = \frac{1}{4}x^4 \Big|_{0}^{2} = 4$

$\int_{0}^{b} x^3\,dx = \frac{1}{2}\int_{0}^{2} x^3 = \frac{1}{2}(4) = 2$

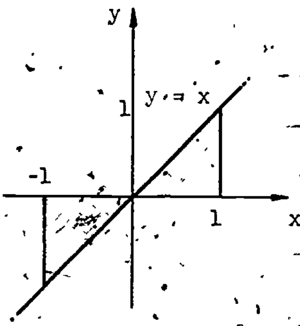
$\therefore\ \frac{1}{4}b^4 = 2$ and $b = \sqrt[4]{8} \approx 1.68$



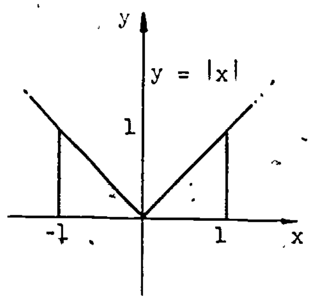

3. (a) $\int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$

(b) $\int_{-1}^{1} |x|dx = \int_{-1}^{0} (-x)dx + \int_{0}^{1} x\,dx = -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$

(c) Same as (b)



(d) Same as (b) and (c).



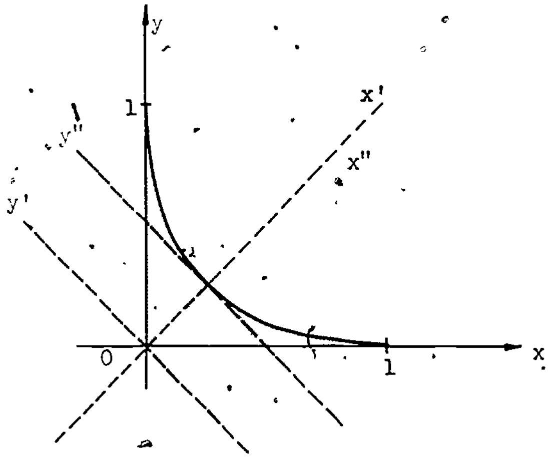

4. $\sqrt{x} + \sqrt{y} = 1$

$\therefore\ y = 1 - 2\sqrt{x} + x$

$$A = \int_0^1 (1 - 2\sqrt{x} + x)\,dx$$

$$= x - \frac{4}{3}x^{3/2} + \frac{x^2}{2}\Big|_0^1$$

$$= 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}$$




If the class has done translations and rotations, this is an excellent place to use them. Otherwise, disregard the following statements.

This is the equation of a parabola for $0 \le x \le 1$ and $0 \le y \le 1$. This can be seen by first simplifying $\sqrt{x} + \sqrt{y} = 1$:

$$x + 2\sqrt{xy} + y = 1$$

$$4xy = 1 + x^2 + y^2 - 2x - 2y + 2xy$$

$$x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$$

Since $B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$, the graph is a parabola. We can rotate the axis $45^\circ$ by substituting

$$x = x' \cos\theta - y' \sin\theta = \frac{1}{\sqrt{2}}(x' - y')$$

$$y = x' \sin\theta + y' \cos\theta = \frac{1}{\sqrt{2}}(x' - y')$$

in the equation $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$

$$\frac{1}{2}(\cancel{x'^2} + \cancel{y'^2} - \cancel{2x'y'} + \cancel{x'^2} + y'^2 + 2x'\cancel{y'} - \cancel{2x'^2} + 2y'^2) - \frac{2}{\sqrt{2}}(x' - \cancel{y'} + x' + \cancel{y'}) + 1 = 0$$

$$2y'^2 = 2\sqrt{2}\,x' - 1$$

$$y'^2 = \sqrt{2}\left(x' - \frac{\sqrt{2}}{4}\right)$$

Now translating the $x'$, $y'$-axes, by substituting

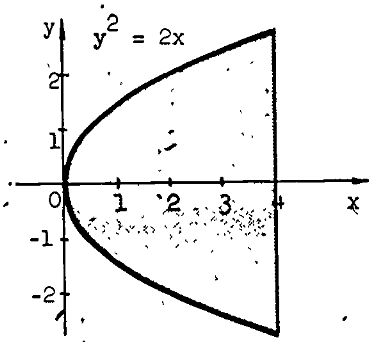
$$x' = x'' + \frac{\sqrt{2}}{4} \quad \text{and} \quad y' = y'',$$

we have

$$y''^2 = \sqrt{2}\, x''$$

which is certainly a parabola.

5. Since $y^2 = 2x$ is symmetric with respect to the x-axis, the area of the entire region may be expressed as double that of the part above the x-axis. Therefore, we have




$$A = 2\int_0^4 \sqrt{2x}\,dx = 2\sqrt{2}\int_0^4 x^{1/2}dx$$

$$= 2\sqrt{2}\cdot\frac{2}{3}x^{3/2}\Big|_0^4$$

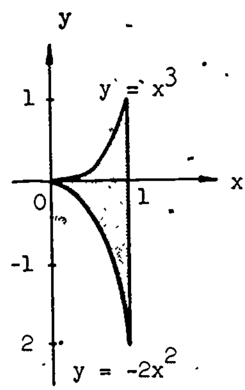
$$= \frac{4\sqrt{2}}{3}(4)^{3/2} = \frac{32\sqrt{2}}{3}$$

6. $$A = \int_0^1 x^3\,dx + \int_0^1 -(-2x^2)dx$$

$$= \int_0^1 x^3\,dx + \int_0^1 2x^2\,dx$$

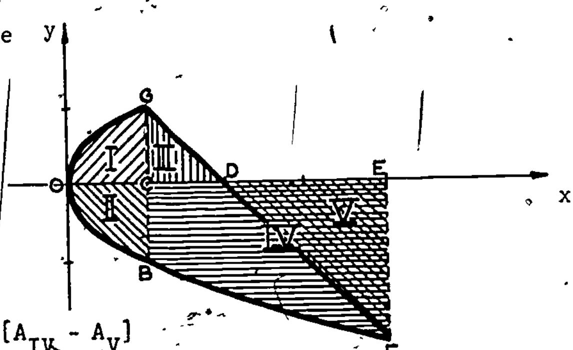
$$= \int_0^1 (x^3 + 2x^2)dx = \left(\frac{x^4}{4} + \frac{2}{3}x^3\right)\Big|_0^1$$

$$= \frac{1}{4} + \frac{2}{3} = \frac{11}{12}$$




7. (a) Subregions are defined by the following:

Region I : OCG
Region II : OCB
Region III : GCD
Region IV : BCEF
Region V : DEF




$\therefore$ Area $= A_I + A_{II} + A_{III} + [A_{IV} - A_V]$

(b) Subregions are defined by the following:

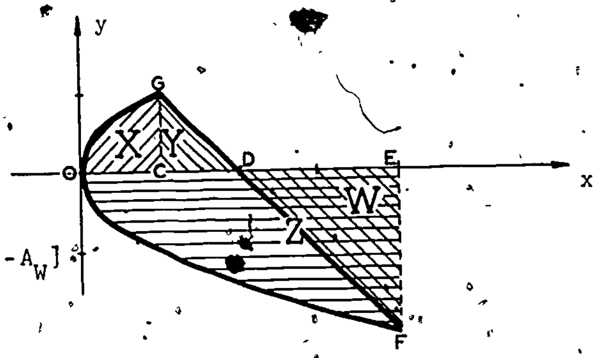
Region X: OCG

Region Y: GCD

Region Z: OEF

Region W: DEF

$\therefore$ Area $= A_X + A_Y + [A_Z - A_W]$




(c) Simplifying part (a)

$$A = \int_0^1 \sqrt{x}\,dx + \int_0^1 -(-\sqrt{x})\,dx + \int_1^2 (-x+2)\,dx + \int_1^4 -(-\sqrt{x})\,dx - \int_2^4 -(-x+2)\,dx$$

$$= 2\int_0^1 \sqrt{x}\,dx + \int_1^4 \sqrt{x}\,dx + \int_1^2 (-x+2)\,dx + \int_2^4 (-x+2)\,dx$$

$$= 2\int_0^1 \sqrt{x}\,dx + \int_1^4 \sqrt{x}\,dx + \int_1^4 (-x+2)\,dx$$

$$= 2\int_0^1 \sqrt{x}\,dx + \int_1^4 [(-x+2) + \sqrt{x}]\,dx$$

Simplifying part (b)

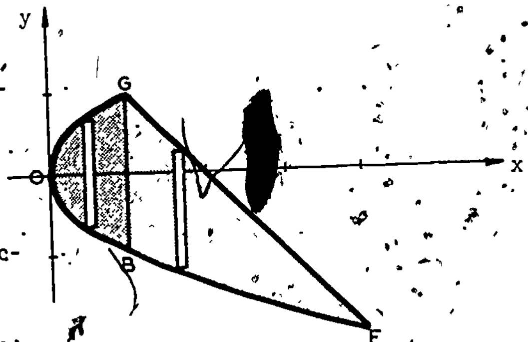
$$A = \int_0^1 \sqrt{x}\,dx + \int_1^2 (-x+2)\,dx + \int_0^4 -(-\sqrt{x})\,dx - \int_2^4 -(-x+2)\,dx$$

$$= \int_0^1 \sqrt{x}\,dx + \int_0^4 \sqrt{x}\,dx + \int_1^2 (-x+2)\,dx + \int_2^4 (-x+2)\,dx$$

$$= \int_0^1 \sqrt{x}\,dx + \int_0^1 \sqrt{x}\,dx + \int_1^4 \sqrt{x}\,dx + \int_1^4 (-x+2)\,dx$$

$$= 2\int_0^1 \sqrt{x}\,dx + \int_1^4 [(-x+2) + \sqrt{x}]\,dx$$

In order to see the relationship between this integral expression for the area, divide the region into two parts. If we add $n$ rectangles in region OGB, we have each rectangle with a height of $\sqrt{x} - (-\sqrt{x})$ or $(2\sqrt{x})$.




If we add $n$ rectangles in region BEF, we have each rectangle with a height of $(-x + 2) - (-\sqrt{x})$ or $(-x + 2 + \sqrt{x})$.

(d) $A = 2\int_0^1 \sqrt{x}\,dx + \int_1^4 (-x + 2 + \sqrt{x})dx$

$= \frac{4}{3}x^{3/2}\Big|_0^1 + \left(-\frac{x^2}{2} + 2x + \frac{2}{3}x^{3/2}\right)\Big|_1^4$

$= [\frac{4}{3} - 0] + [(-8 + 8 + \frac{16}{3}) - (-\frac{1}{2} + 2 + \frac{2}{3})]$

$= \frac{4}{3} + \frac{16}{3} - \frac{13}{6}$

$= \frac{27}{6} = \frac{9}{2}$

8. (a) (i) Area of Region I $= \int_0^2 (2x - x^2)dx$

(ii) Area of Region II $= \int_{-1}^0 -(2x - x^2)dx$

(iii) Area of Region III $= \int_2^3 -(2x - x^2)dx$

(iv) Area of Region IV $= \int_{-1}^3 -(-3)dx$

(b) Area of region bounded by $y = 2x - x^2$ and $y = -1$ can be expressed as follows:

A = Area of Region I + [Area of Region IV - Area of Region II - Area of Region III]

$= \int_0^2 (2x - x^2)dx + \int_{-1}^3 3\,dx + \int_{-1}^0 (2x + x^2)dx + \int_2^3 (2x - x^2)dx$

Combining integrals 1, 3, and 4, we have

$$A = \int_{-1}^3 (2x - x^2)dx + \int_{-1}^3 3\,dx$$

$$= \int_{-1}^3 (2x - x^2 + 3)dx$$

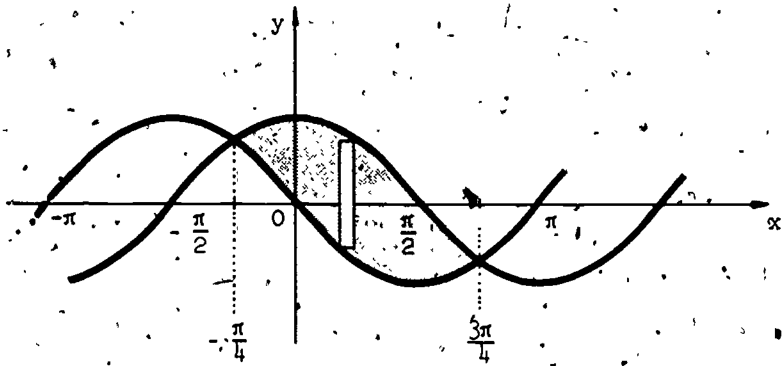
[Note that $(2x - x^2) - (-3)$ or $(2x - x^2 + 3)$ is the height of each rectangle.]

(c) $A = \int_{-1}^{3}(2x - x^2 + 3)dx$

$$= (x^2 - \frac{x^3}{3} + 3x)\Big|_{-1}^{3} = (9 - 9 + 9) - (1 + \frac{1}{3} - 3) = \frac{32}{3}$$

9. (a) $\begin{cases} y = \cos x \\ y = -\sin x \end{cases}$

Intersections for $|x| \leq \pi$ are $x = \frac{3\pi}{4}$ and $x = -\frac{\pi}{4}$.




$$A = \underbrace{\int_{-\pi/4}^{\pi/2}\cos x\,dx - \int_{-\pi/4}^{0}(-\sin x)dx}_{\text{area above x-axis}} + \underbrace{\int_{0}^{3\pi/4}(-\sin x)dx - \int_{\pi/2}^{3\pi/4}(-\cos x)dx}_{\text{area below x-axis}}$$

Combining integrals, we have

$$A = \int_{-\pi/4}^{3\pi/4}\cos x\,dx + \int_{-\pi/4}^{3\pi/4}\sin x\,dx = \int_{-\pi/4}^{3\pi/4}(\cos x + \sin x)dx$$

[Note that height of rectangle is $\cos x - (-\sin x)$ or $(\cos x + \sin x)$.]

$$\therefore A = (\sin x - \cos x)\Big|_{-\pi/4}^{3\pi/4} = [\sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}] - [\sin(-\frac{\pi}{4}) - \cos(-\frac{\pi}{4})]$$

$$= [\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}] - [-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}] = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

(b) (i) $$\int_{-\pi/4}^{3\pi/4} \cos x \, dx = \sin x \Big|_{-\pi/4}^{3\pi/4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

(ii) $$\int_{-\pi/4}^{3\pi/4} (-\sin x) dx = \cos x \Big|_{-\pi/4}^{3\pi/4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

(iii) $$\int_{-\pi/4}^{3\pi/4} (\cos x - \sin x) dx = (\sin x + \cos x) \Big|_{-\pi/4}^{3\pi/4}$$

$$= \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0$$

(iv) These are signed areas. The answer to (iii) is zero which means that the area above the axis is equal to the area below the axis.

10. (a) Find $\int_{-a}^{a} f$ when $f$ is an odd function.

Since $f(-x) = -f(x)$ the area bounded by $x = -a$, $f(x)$ and $x = 0$ has the same magnitude but opposite sign when compared to the area bounded by $x = a$, $f(x)$ and $x = 0$.

Since $$\int_{-a}^{0} f = -\int_{0}^{a} f$$

then $$\int_{-a}^{a} f = \int_{-a}^{0} f + \int_{0}^{a} f$$

$$= -\int_{0}^{a} f + \int_{0}^{a} f$$

$$= 0$$

(b) When $f$ is an even function, $f(x) = f(-x)$. The area bounded by $x = -a$, $f(x)$ and $x = 0$ is numerically equal to the area bounded by $x = a$, $f(x)$ and $f(0)$.

Since $$\int_{-a}^{0} f = \int_{0}^{a} f$$

then $$\int_{-a}^{a} f = \int_{-a}^{0} f + \int_{0}^{a} f = 2\int_{0}^{a} f$$

(c) $\int_{-5}^{5}(x^3 - 3x)\sin x^2\,dx$

$f : x \to (x^3 - 2x)$ is an odd function.

$g : x \to \sin x^2$ is an even function.

The product $f \cdot g$ is an odd function.

Thus $\int_{-a}^{a} f \cdot g = 0$ by part (a).

11. If $F' = f$ and $G' = g$ and $f(x) \le g(x)$ for $a \le x \le b$, then $F(b) - F(a) \le G(b) - G(a)$. By 7 - 2 - (5). If $f(x) \le g(x)$ for $a \le x \le b$ then $\int_a^b f \le \int_a^b g$. By the Fundamental Theorem of Calculus,

$$\int_a^b f = F(b) - F(a)$$

and

$$\int_a^b g = G(b) - G(a).$$

Thus $$F(b) - F(a) \le G(b) - G(a).$$

12. $\int_a^b f = F(b) - F(a)$

Verify (5)(a): $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$

Let $h(x) = f(x) + g(x)$ and $H(x) = F(x) + G(x)$.

$$\begin{aligned}\int_a^b h(x)dx &= H(b) - H(a)\\ &= (F(b) + G(b)) - (F(a) + G(a))\\ &= (F(b) - F(a)) + (G(b) - G(a))\\ &= \int_a^b f(x)dx + \int_a^b g(x)dx\end{aligned}$$

Verify (5)(b): $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

$$\int_a^b c\, f(x)dx = cF(a) - cF(b)$$

$$= c(F(a) - F(b))$$

$$= c\int_a^b f(x)dx$$

Verify (5)(c): $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad a \le c \le b$

$$\int_a^c f(x)dx = F(c) - F(a)$$

$$\int_c^b f(x)dx = F(b) - F(c)$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = (F(c) - F(a)) + (F(b) - F(c))$$

$$= (F(b) - F(a)) + (F(c) - F(c))$$

$$= \int_a^b f(x)dx$$

13. $F(x) = \int_x^1 f$ where $f : x \to e^x$

(a) $F(1) = 0$

(b) Since $\int_a^b f = -\int_b^a f$ then $F(x) = -\int_1^x f$

$$= -(e^x - e^1)$$

$$= e - e^x$$

(c) $F'(x) = 0 - e^x = -e^x$

(d) If $G(x) = \int_x^b g$ then $G'(x) = -g(x)$

Recall that the Area Theorem stated that when

$$F(x) = \int_a^x f$$

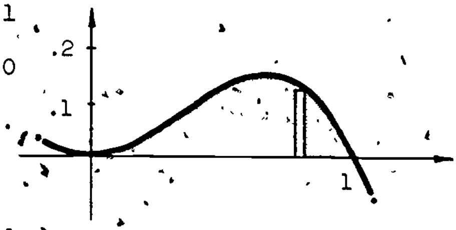
then $$F'(x) = f(x).$$

The integral $G(x) = \int_x^a g$ must be written as an integral from $a$ to $x$, rather than from $x$ to $a$.

We have defined $\int_a^b f = -\int_b^a f$ thus $\int_x^a g = -\int_a^x g = \int_a^x -g$.

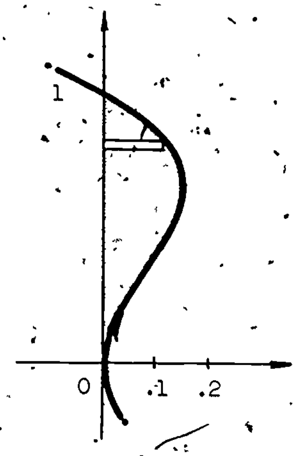
Thus $G'(x) = -g(x)$.

14. (a) $A = \int_0^1 (x^2 - x^3)dx = (\frac{x^3}{3} - \frac{x^4}{4})\Big|_0^1$




Therefore $A = (\frac{1}{3} - \frac{1}{4}) - 0 = \frac{1}{12}$.

(b) It is intuitively obvious that this area is also $\frac{1}{12}$. This problem illustrates a type of problem not discussed in the text. Since it would be very difficult to set up the area integral so that the rectangles are summed along the x-axis, it is possible to sum rectangles along the y-axis. Then we would have




$$A = \int_0^1 (y^2 - y^3)dy = (\frac{y^3}{3} - \frac{y^4}{4})\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

## Solutions Exercises 7-6

1. $\int (x^2 + 1)dx = \frac{x^3}{3}\ x$

2. $\int (\frac{1}{x^2} + x + x^4)dx = -\frac{1}{x} + \frac{x^2}{2} + \frac{x^5}{5}$

3. $\int 8\sqrt{x}\, dx = 8\int x^{1/2}dx = \frac{16}{3} x^{3/2}$

4. $\int (x^2 - \sqrt{x})dx = \frac{x^3}{3} - \frac{2x^{3/2}}{3}$

5. $\int (\frac{1 - x}{x})dx = \int (\frac{1}{x} - 1)dx = \log_e x - x$

6. $\int \sin 3x\, dx = -\frac{1}{3} \cos 3x$

7. $\int \cos(2x - 5)dx = \frac{1}{2} \sin(2x - 5)$

8. $\int (-\sin 2x)dx = -\int \sin 2x\, dx = -(-\frac{1}{2} \cos 2x) = \frac{1}{2} \cos 2x$

9. $\int [-\cos(3x - 1)]dx = -\int \cos(3x - 1)dx = -\frac{1}{3} \sin(3x - 1)$

10. $\int \frac{4}{3} \cos 3x\, dx = \frac{4}{3}\int \cos 3x\, dx = \frac{4}{9} \sin 3x$

11. $\int 2 \sin x \cos x\, dx = \int \sin 2x = -\frac{1}{2} \cos 2x$

12. $\int (3 \sin 2x - 6 \cos 3x)dx = -\frac{3}{2} \cos 2x - 2 \sin 3x$

13. $\int e^{2x}\, dx = \frac{1}{2} e^{2x}$

14. $\int e^{x/3}\, dx = 3e^{x/3}$

15. $\int (e^x + e^{-x})^2\, dx = \int (e^{2x} + 2 + e^{-2x})dx = \frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x}$

16. $\int x^2e^x dx = x^2e^x - 2\int xe^x dx$

$= x^2e^x - 2(xe^x - e^x)$

$= x^2e^x - 2xe^x + 2e^x$

$= e^x(x^2 - 2x + 2)$

17. $\int x^3e^x dx = x^3e^x - 3\int x^2e^x dx$

$= x^3e^x - 3[e^x(x^2 - 2x + 2)]$

$= e^x(x^3 - 3x^2 + 6x - 6)$

18. $\int x^4e^x dx = x^4e^x - 4\int x^3e^x dx$

$= x^4e^x - 4[e^x(x^3 - 3x^2 + 6x - 6)]$

$= e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)$

19. $\int x^2 \log_e x\, dx = \frac{x^3}{3}(\log_e x - \frac{1}{3})$

20. $\int x^3 \log_e x\, dx = \frac{x^4}{4}(\log_e x - \frac{1}{4})$

21. $\int x^4 \log_e x\, dx = \frac{x^5}{5}(\log_e x - \frac{1}{5})$

22. $\int x^2 \sin x\, dx = -x^2 \cos x + 2\int x \cos x\, dx$

$= -x^2 \cos x + 2[x \sin x + \cos x]$

$= 2x \sin x + (2 - x^2)\cos x$

23. $\int x^3 \sin x\, dx = -x^3 \cos x + 3\int x^2 \cos x\, dx$

$= -x^3 \cos x + 3[x^2 \sin x - 2\int x \sin x\, dx]$

$= -x^3 \cos x + 3[x^2 \sin x - 2(-x \cos x + \sin x)]$

$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$

$= (6x - x^3)\cos x + (3x^2 - 6)\sin x$

24. $\int e^{3x} \sin 4x\, dx = \frac{e^{3x}}{25}(3 \sin 4x - 4 \cos 4x)$

25. $$\int e^{x/2}\cos\frac{3x}{2}\,dx = \frac{e^{x/2}}{\frac{10}{4}}\left(\frac{3}{2}\sin\frac{3x}{2} + \frac{1}{2}\cos\frac{3x}{2}\right)$$

$$= \frac{2}{5}e^{x/2}\cdot\frac{1}{2}\left(3\sin\frac{3x}{2} + \cos\frac{3x}{2}\right)$$

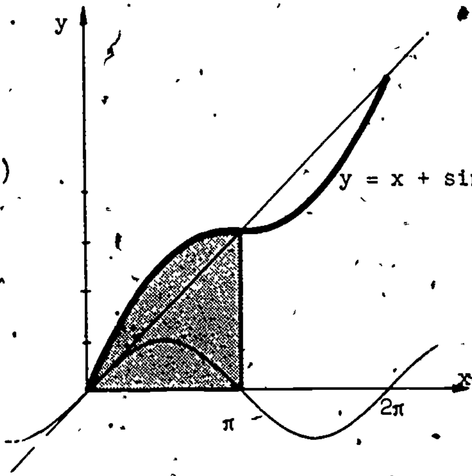
$$= \frac{1}{5}e^{x/2}\left(3\sin\frac{3x}{2} + \cos\frac{3x}{2}\right)$$

26. $$\int_0^{\pi}(x + \sin x)dx$$

$$= \left(\frac{x^2}{2} - \cos x\right)\Big|_0^{\pi}$$

$$= \left(\frac{\pi^2}{2} + 1\right) - (0 - 1)$$

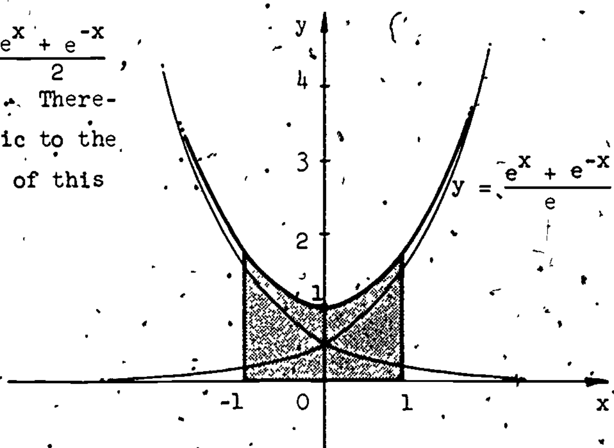
$$= \frac{\pi^2}{2} + 2 \approx 6.9$$




27. $$\int_0^{2\pi}(x + \sin x)dx = \left(\frac{x^2}{2} - \cos x\right)\Big|_0^{2\pi} = (2\pi^2 - 1) - (0 - 1) = 2\pi^2 \approx 2.35$$

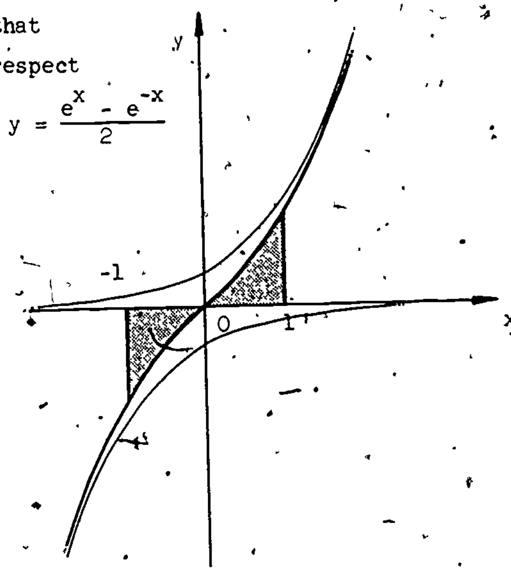
28. $$\int_{-1}^{1}\frac{e^x + e^{-x}}{2}\,dx$$

This function $x \to \frac{e^x + e^{-x}}{2}$, is an even function. Therefore, it is symmetric to the y-axis. Making use of this symmetry, we have




$$\int_{-1}^{1}\frac{e^x + e^{-x}}{2} = 2\left[\frac{1}{2}\right]\int_0^1 (e^x + e^{-x})dx = (e^x - e^{-x})\Big|_0^1 = e - \frac{1}{e} \approx 2.35$$

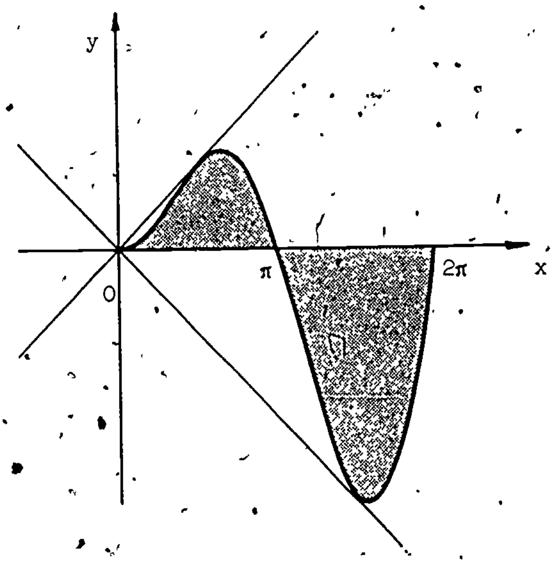
29. (a) $\int_{-1}^{1} \frac{e^x - e^{-x}}{2} = 0$. Here we have an odd function which means that the curve is symmetric with respect to the origin.




30. (a) $\int_{0}^{2\pi} x \sin x \, dx = (-x \cos x + \sin x) \Big|_{0}^{2\pi}$

$= (-2\pi + 0) - (0 + 0)$

$= -2\pi$



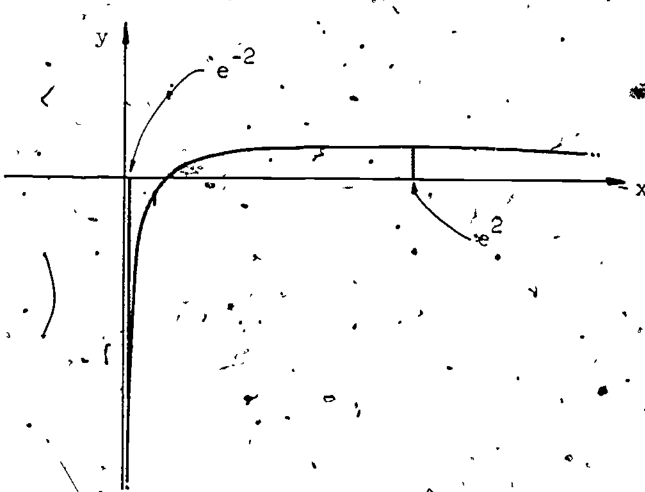
31. (a) $\int_{1/e^2}^{e^2} \frac{\log_e x}{\sqrt{x}}\,dx = \int_{1/e^2}^{e^2} x^{(-1/2)}\log_e x\,dx$

$$= \frac{x^{1/2}}{\frac{1}{2}}\left(\log_e x - \frac{1}{\frac{1}{2}}\right)\Bigg|_{1/e^2}^{e^2}$$

$$= 2\sqrt{x}(\log_e x - 2)\Bigg|_{1/e^2}^{e^2}$$

$$= [2e(2 - 2)] - [\tfrac{2}{e}(-2 - 2)]$$

$$= 0 + \frac{8}{e} = \frac{8}{e} \approx 2.96$$

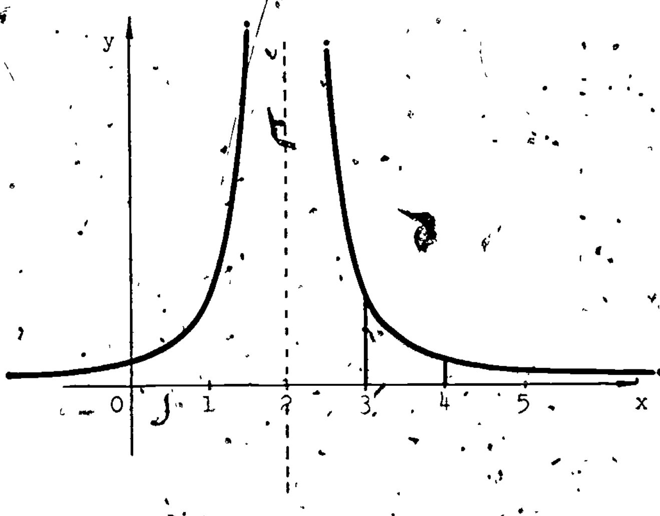



32. $A = \int_3^4 \frac{1}{(x - 2)^2}\,dx$

$= -(x - 2)^{-1}\Big|_3^4$

$= -\frac{1}{2} + 1$

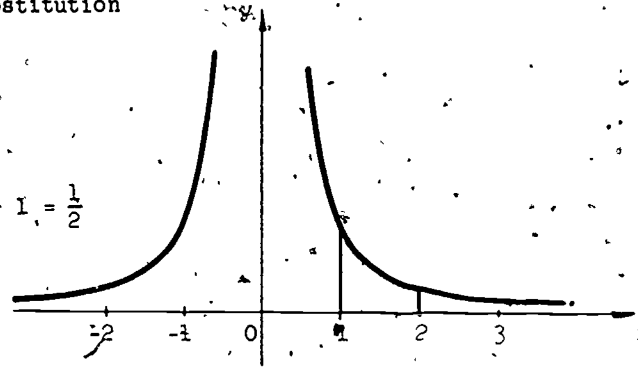
$= \frac{1}{2}$



Replace $x - 2$ by $x$ (i.e., $x$ by $x + 2$). This linear substitution leads to

$$A_{L.S.} = \int_1^2 \frac{1}{x^2}\,dx$$

$$= -x^{-1}\Big|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$



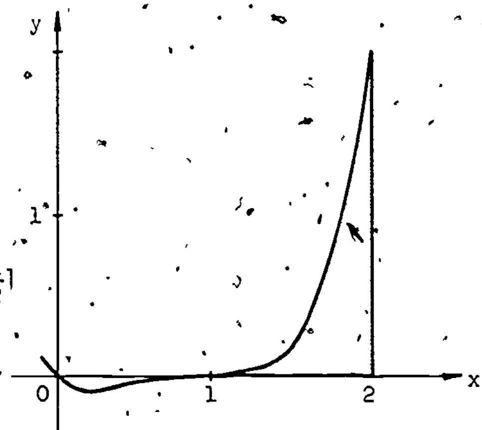

33) $$A = \int_1^2 x(x - 1)^3$$

$$= \int_1^2 (x^4 - 3x^3 + 3x^2 - x)\,dx$$

$$= \left(\frac{x^5}{5} - \frac{3}{4}x^4 + x^3 - \frac{x^2}{2}\right)\Big|_1^2$$

$$= \left[\frac{32}{5} - 12 + 8 - 2\right] - \left[\frac{1}{5} - \frac{3}{4} + 1 - \frac{1}{2}\right]$$

$$= \left[\frac{2}{5}\right] + \left[\frac{1}{20}\right] = \frac{9}{20}$$

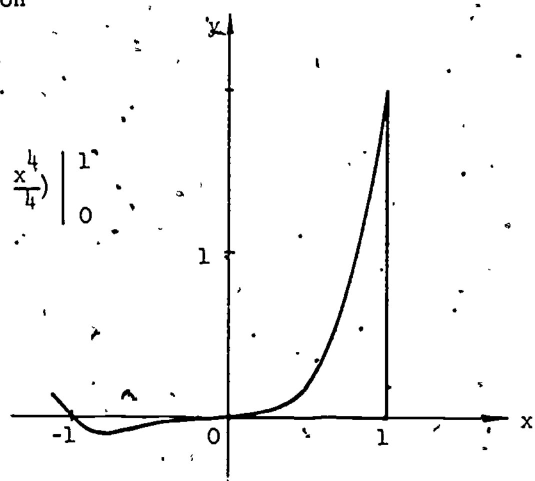



Replace $x - 1$ by $x$ (i.e., $x$ by $x + 1$). This linear substitution leads to

$$A_{L.S.} = \int_0^1 (x + 1)x^3\,dx$$

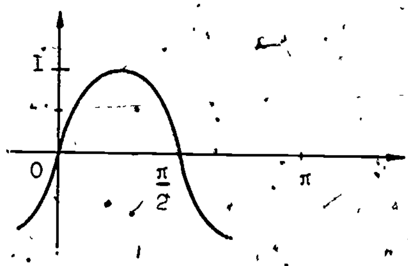
$$= \int_0^1 (x^4 + x^3)\,dx = \left(\frac{x^5}{5} + \frac{x^4}{4}\right)\Big|_0^1$$

$$= \frac{1}{5} + \frac{1}{4} = \frac{9}{20}$$



34. $A = \int_0^{\pi/2} \sin 2x \, dx$

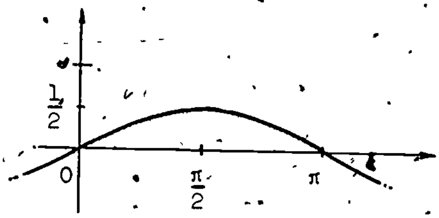
$= -\frac{1}{2}\cos 2x \Big|_0^{\pi/2} = 1$




Substitute $x$ for $2x$ (i.e., let $x = \frac{x}{2}$).

$\therefore A_{L.S.} = \frac{1}{2}\int_0^{\pi} \sin x \, dx$

$= -\frac{1}{2}\cos x \Big|_0^{\pi} = 1$

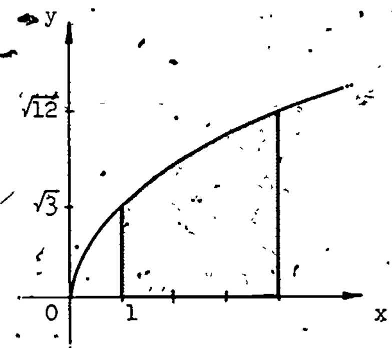



35. $A = \int_1^4 \sqrt{3x} \, dx$

$= \sqrt{3}\int_1^4 x^{1/2} \, dx$

$= \sqrt{3} \cdot \frac{2}{3} x^{3/2} \Big|_1^4$

$= \frac{2\sqrt{3}}{3}(8 - 1) = \frac{14}{3}\sqrt{3}$




Substitute $x$ for $3x$ (i.e., let $x = \frac{x}{3}$).

$\therefore A_{L.S.} = \frac{1}{3}\int_3^{12} \sqrt{x} \, dx$

$= \frac{1}{3} \cdot \frac{2}{3} x^{3/2} \Big|_3^{12}$

$= \frac{2}{9}(12^{3/2} - 3^{3/2}) = \frac{2}{9}(3^{3/2} \cdot 8 - 3^{3/2}) = \frac{14}{3}\sqrt{3}$

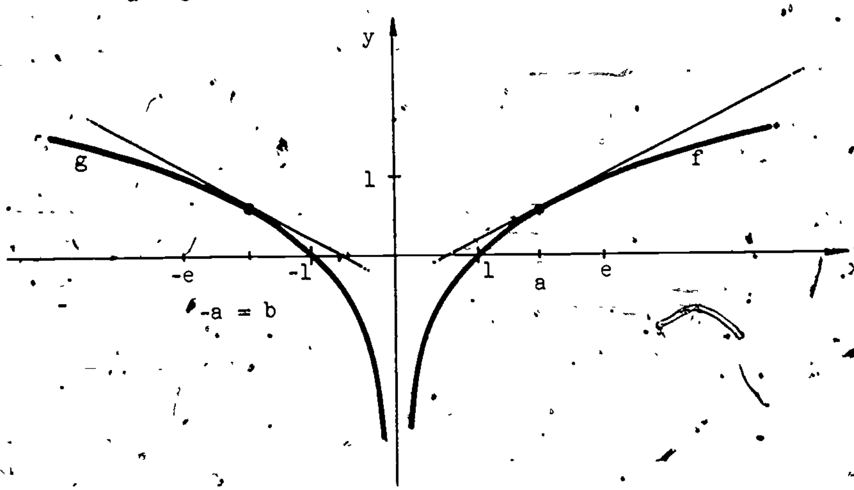
36. (a) An intuitive treatment of this problem leads one to compare the graphs of

$$f : x \to \log_e (x) , x > 0$$

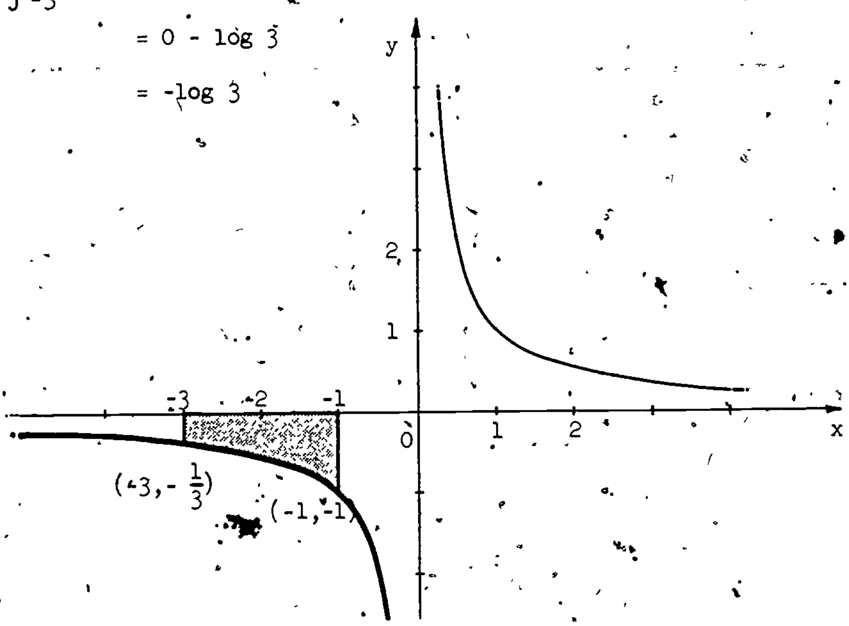
and $$g : x \to \log_e(-x) , x < 0.$$

The tangent line to $f$ at $a > 0$ has the slope $D \log_e a = \frac{1}{a}$.

Because of symmetry the tangent line to $g$ at $b = -a$ is the opposite of the slope of $f$ at $a > 0$. Thus the slope of $g$ at $b$ is $\frac{1}{-a} = \frac{1}{b}$. It follows then that $D \log_e(-x) = \frac{1}{x}$.




(b) $$\int_{-3}^{-1} \frac{1}{x}\, dx = \log_e (-(-1)) - \log(-(-3))$$
$$= 0 - \log 3$$
$$= -\log 3$$



37. (a) By the Fundamental Theorem of Calculus $f$ must have no gaps on the interval $[a,b]$. In the case of $f : x \to \frac{1}{x}$, $f$ on the interval $[-1,1]$ has a gap at $x = 0$. Thus we cannot apply the Fundamental Theorem of Calculus.

(b) $$\lim_{n \to \infty} \int_{1/n}^{1} \frac{1}{x}\, dx = \lim_{n \to \infty} \left(\log_e 1 - \log_e \frac{1}{n}\right)$$

$$= \lim_{n \to \infty} \left(\log_e 1 + \log_e n\right)$$

$$= 0 + \infty$$

$$= \infty$$

(c) The area assigned to the region bounded by $y = \frac{1}{x}$, $x \neq 0$, the y-axis, the x-axis and $x = 1$ is well defined so long as $x \neq 0$. This area has a finite value so long as $x$ has a finite value. Should we take an integral with $x = 0$ as an endpoint then the integral is undefined.

(d) Since $f$ has a gap on the interval $-1 \leq x \leq 1$ we cannot apply the Fundamental Theorem of Calculus. We can, however, try the mechanics of integration. We find that our answer is zero.

$$\int_{-1}^{1} \frac{1}{x}\, dx = \log (1) - \log (-(-1))$$

$$= 0 - 0$$

$$= 0$$

This might be consistent to the extent that we are integrating over two areas with the "same" magnitude but opposite in sign. Thus they cancel each other out.

On the other hand the inconsistency comes in our effort to write the statement needed to justify cancelling out areas, namely:

$$\infty + (-\infty) = 0.$$

Teacher's Commentary

Chapter 8

# DIFFERENTIATION THEORY AND TECHNIQUE

A promise was implicit in the earlier Teacher's Commentary statement:

> "... In this version the special functions are first studied in some detail with the aid of the calculus, which is introduced intuitively, and later [Chapter 8] the general techniques of Calculus are developed and applied to a wide class of functions."

In this chapter we deliver on that promise as we discuss concepts and techniques pertaining to the differentiation of algebraic combinations of functions.

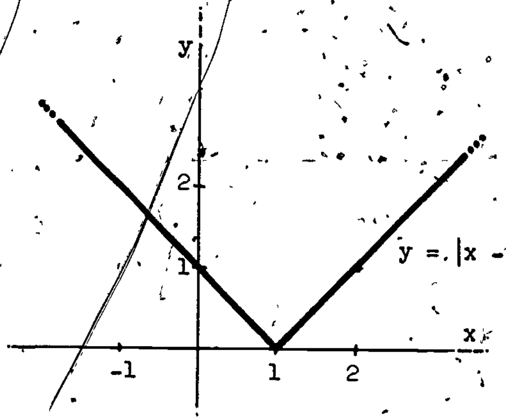
We begin by reviewing the derivatives of some typical functions studied earlier and abbreviating the language we use. For example, the derivative of the function $f : x \to x^2$ is the function $f' : x \to 2x$. The value of the derivative of $f$ at $(x,f(x))$ is $f'(x) = 2x$; that is, $D(x^2) = 2x$. We shorten this by saying (in Example 8-1a), "the derivative of $x^2$ is $2x$."

A more analytical approach to the fundamental ideas and basic theorems of Chapter 8 can be found in Appendix 7.

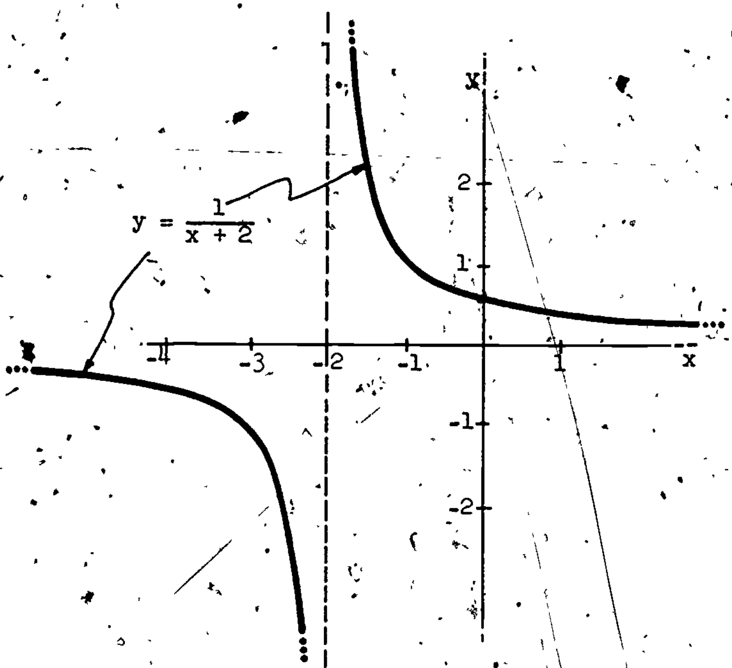
| Chapter 8 | | Appendix 7 | |
|---|---|---|---|
| 8-1. | Differentiability | A7-1. | Completeness of the Real Number System. The Separation Axiom |
| 8-2. | Continuous Functions | A7-2. | The Extreme Value and Intermediate Value Theorems |
| 8-3. | The Mean Value Theorem | A7-3. | The Mean Value Theorem |
| 8-4. | Applications of the Mean Value Theorem | A7-4. | Applications of the Mean Value Theorem for Continuous Functions |

## Solutions Exercises 8-1

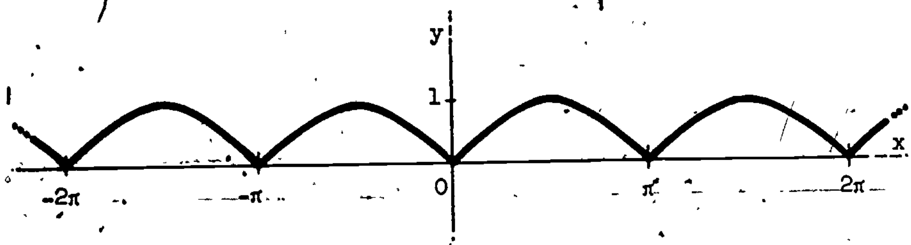
1. $f$ is not differentiable for $x = 1$, since the graph of $f$ is the graph of $|x|$ translated one unit to the right. It has a corner at $x = 1$.



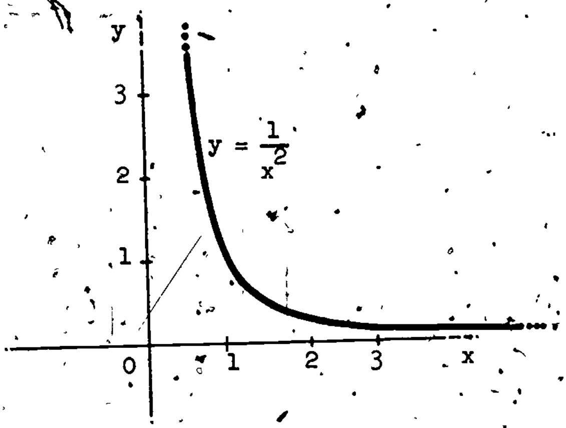

2. $f$ is not differentiable at $x = -2$ since $f(-2)$ is undefined and hence, $f$ is not continuous at $x = -2$.



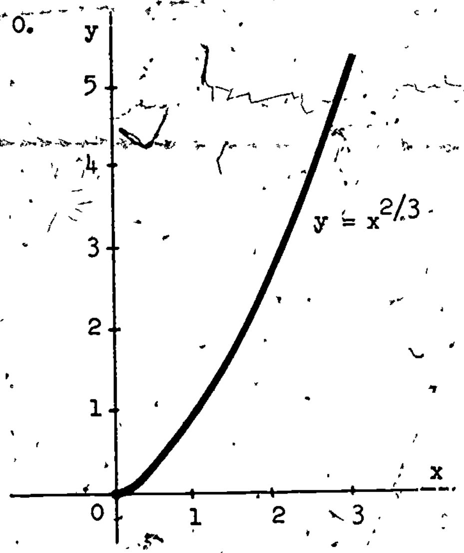
3. f is not differentiable at $x = n\pi$, for any integer n since the graph of f has corners at all such points.



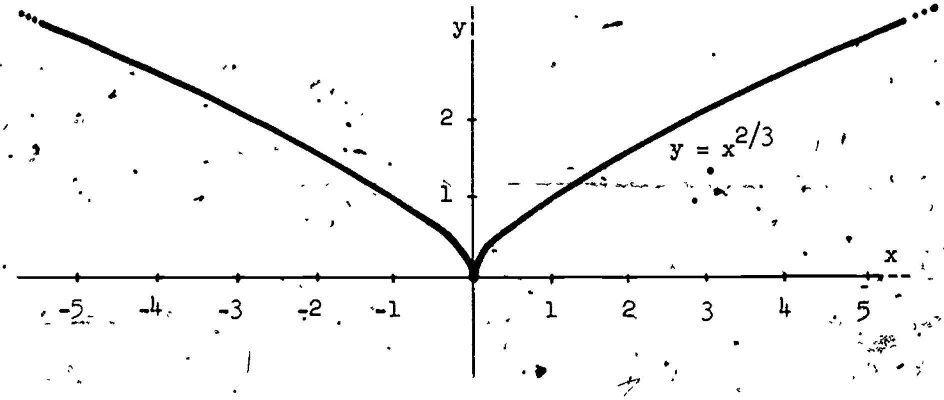

4. f is not differentiable at $x = 0$ because $f(0)$ is undefined and hence, f is not continuous at $x = 0$.




5. f is not differentiable for any negative values of x because $f(x)$ is undefined for $x < 0$.



6. f is not differentiable for $x = 0$ because $D(x^{2/3}) = \frac{2}{3}x^{-1/3} = \frac{2}{3}\cdot\frac{1}{\sqrt[3]{x}}$ which does not exist for $x = 0$.




7. (a) $f(\frac{1}{n\pi}) = \sin n\pi = 0$

(b) $f\left(\frac{1}{\frac{\pi}{2}}\right) = \sin\frac{\pi}{2} = 1$

$f\left(\frac{1}{\frac{5\pi}{2}}\right) = \sin\frac{5\pi}{2} = 1$

$f\left(\frac{1}{\frac{9\pi}{2}}\right) = \sin\frac{9\pi}{2} = 1$

$\vdots$

$f\left(\frac{1}{\frac{(4n+1)\pi}{2}}\right) = \sin\frac{(4n+1)\pi}{2} = \sin(2n\pi + \frac{\pi}{2}) = 1$

(c) $f\left(\frac{1}{\frac{3\pi}{2}}\right) = \sin\frac{3\pi}{2} = -1$

$f\left(\frac{1}{\frac{7\pi}{2}}\right) = \sin\frac{7\pi}{2} = -1$

$f\left(\frac{1}{\frac{11\pi}{2}}\right) = \sin\frac{11\pi}{2} = -1$

$\vdots$

$f\left(\frac{1}{\frac{(4n-1)\pi}{2}}\right) = \sin\frac{(4n-1)\pi}{2} = \sin(2n\pi - \frac{\pi}{2}) = -1$

No; as h approaches zero, f(h) takes all values from -1 to +1 infinitely often.

## Solutions Exercises 8-2

1. $f(-1) = 2$, $f(1) = -2$, and $f$ is continuous on $[-1,1]$ so there must be at least one number $c$ in $[-1,1]$ such that $f(c) = 0$. In fact, $f(0) = 0$.

2. $f$ is not continuous on $[-1,2]$ so the Intermediate Value Theorem does not apply, even though there does exist a number $c$ in $[-1,2]$ such that $f(c) = \frac{3}{2}$, namely $c = \frac{3}{2}$.

3. $f(-1) = 2$, $f(1) = -2$, and $f$ is continuous on $[-1,1]$ so there must be at least one number $c$ in $[-1,1]$ such that $f(c) = 1$.

4. $f$ is not continuous on $[-1,1]$ because it is not continuous at $x = 0$, so the Intermediate Value Theorem does not apply.

5. $f(0) = 0$, $f(\frac{\pi}{2}) = 1$, and $f$ is continuous on $[0, \frac{\pi}{2}]$, so there must be at least one number $c$ in $[0, \frac{\pi}{2}]$ such that $f(c) = \frac{1}{2}$. In fact, $f(\frac{\pi}{6}) = \frac{1}{2}$.

6. $f(0) = 0$, $f(\frac{\pi}{2}) = 1$, but $C = 2$ is not between $0$ and $1$ so the Intermediate Value Theorem does not apply.

7. $f$ is discontinuous at $x = 0$ so the Intermediate Value Theorem does not apply.

8. $m = f(1) = -2$; $M = f(-1) = 2$

9. $m = f(1) = 0$; $M = f(0) = f(2) = 1$

10. $m = f(1) = f(-2) = -2$; $M = f(-1) = f(2) = 2$

11. $f$ has no minimum nor maximum on $[-1,1]$, since $\lim_{x \to 0} \frac{1}{x}$ is infinitely negative for $x < 0$ and infinitely positive for $x > 0$.

12. $m = f(0) = 0$; $M = f(\frac{\pi}{2}) = 1$

13. $m = 0 = f(x)$ for all $x < 0$ in $[-1,1]$
    $M = 1 = f(x)$ for all $x \geq 0$ in $[-1,1]$

1. $(\frac{1}{2}, \frac{3}{4})$

2. $Q(4,16)$

3. $x = \sqrt[3]{\frac{5}{4}}$

4. Yes. The elevation is a continuous function $f$ of the distance traveled. Assuming moreover, that $f$ is differentiable, we may apply the Mean Value Theorem and conclude that there must be some number $c$, $0 < c < 100$, such that

$$f'(c) = \frac{f(100) - f(0)}{100 - 0}.$$

Expressing elevation in miles, this ratio is $0.01$ or $1\%$. Hence, the slope of the road at some point $c$ is exactly $1\%$.

5. Yes. The speed of the car is a continuous function $f$ of time. The Intermediate Value Theorem guarantees that if $f(t_1) = 0$ and $f(t_2) = 70$ for some values of the time $t$, then there must be some time $t_0$, $t_1 < t_0 < t_2$, such that $f(t_0) = 50$, since $50$ is between $0$ and $70$.

6. (a) Yes. The distance traveled is a continuous function $f$ of time, and the speed of the car is also continuous and is given by the derivative of $f$, assuming $f$ is differentiable. The Mean Value Theorem then guarantees that there is some time $t_0$, $0 < t_0 < 4$, at which

$$f'(t_0) = \frac{f(4) - f(0)}{4 - 0} = \frac{200}{4} = 50.$$

(b) Yes. Assuming the acceleration is continuous, then since it is positive just after $t = 0$ and negative just before $t = 4$, there must be some time $t_0$, $0 < t_0 < 4$, at which it is zero, according to the Intermediate Value Theorem.

7. If $f'(x) = -\frac{1}{x^2} = \frac{1 - \frac{1}{2}}{1 - 2} = -\frac{1}{2}$ then $x = \pm\sqrt{2}$. Hence, there are two points where the tangent is parallel to $\overline{PQ}$: $(\sqrt{2}, \frac{1}{\sqrt{2}})$ is on the same branch of the hyperbola as $\overline{PQ}$, and $(-\sqrt{2}, -\frac{1}{\sqrt{2}})$ is on the other branch.

8. There is no such point. The Mean Value Theorem does not apply because $f$ is discontinuous at $x = 0$ and hence, not differentiable at all points in the interior of the interval $[-1,1]$.

9. There is no such point. The Mean Value Theorem does not apply because $f$ is discontinuous at $x = 0$, the end point of the interval $[0,1]$.

## Solutions Exercises 8-4

1. If $f(a) = f(b) = 0$ where $f$ is differentiable for $a < x < b$ and continuous at $x = a$ and $x = b$, then there is at least one number $c$, $a < c < b$, such that

$$f'(c) = 0.$$

2. By Theorem 8-4a, if $f''(x) > 0$ on the open interval $a < x < b$, then $f'(x)$ increases uniformly on $[a,b]$. Theorem 8-4d guarantees that if $f'$ increases, $f$ is convex. Hence, if $f''(x) > 0$ then $f$ is convex on the appropriate interval.

3. The point $(b,f(b))$ is a relative maximum because $f$ is increasing for $a \leq x < b$ (Theorem 8-4a), and $f$ is decreasing for $b < x < c$ (Theorem 8-4b).

4. Assuming $[f(x) - g(x)]' = f'(x) - g'(x) = 0$, then according to Theorem 8-4c, $F(x) = [f(x) - g(x)]$ is a constant function on $[a,b]$, i.e., $f(x) - g(x) = c$, where $c$ is some constant.

## Solutions Exercises 8-5

1. (a) $y = x^{1/3} - 3x^{-2/5}$

$y' = \frac{1}{3}x^{-2/3} + \frac{6}{5}x^{-7/5}$

(b) $y = x^2 + 2\sin x$

$y' = 2x + 2\cos x$

(c) $y = (3x^2 + 1)(x^4 + 1)$

$= 3x^6 + x^4 + 3x^2 + 1$

$y' = 18x^5 + 4x^3 + 6x$

(d) $y = (1 - 2x)\left(\frac{1}{x^2} + \frac{1}{x}\right)$

$= (1 - 2x)(x^{-2} + x^{-1})$

$= x^{-2} - x^{-1} - 2$

$y' = -2x^{-3} + x^{-2}$

(e) $y = e^x + e^{2x} + \cos x$

$y' = e^x + 2e^{2x} - \sin x$

(f) $y = \sqrt{x} - 3e^{-x}$

$y' = \frac{1}{2}x^{-1/2} + 3e^{-x}$

(g) $y = x + \log_e x^2 - 2\log_e x$

$= x + 2\log_e x - 2\log_e x$

$= x$

$y' = 1$

(h) $y = x^e + e^x$

$y' = ex^{e-1} + e^x$

$= e(x^{e-1} + e^{x-1})$



118-A

2. $f : x \to \sqrt{x} + \frac{1}{x}$, for $0 < x \leq 1$

$f(\frac{1}{2}) = \frac{\sqrt{2}}{2} + 2$

$u : x \to \sqrt{x}$

$u(\frac{1}{2}) = \frac{\sqrt{2}}{2}$

$v : x \to \frac{1}{x}$

$v(\frac{1}{2}) = 2$

$f' : x \to \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$

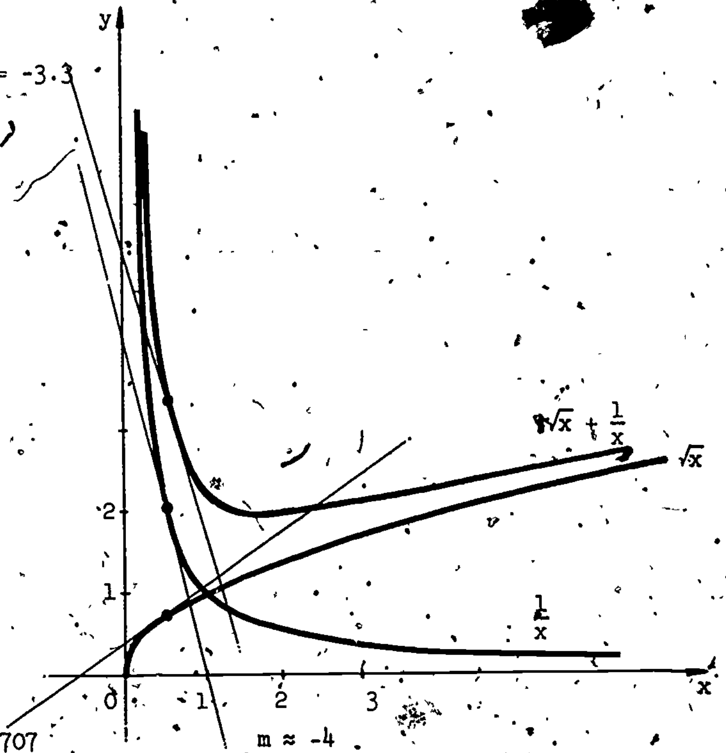
$f'(\frac{1}{2}) = \frac{\sqrt{2}}{2} - 4$

$u' : x \to \frac{1}{2\sqrt{x}}$

$u'(\frac{1}{2}) = \frac{\sqrt{2}}{2}$

$v' : x \to -\frac{1}{x^2}$

$v'(\frac{1}{2}) = -4$




Tangent lines at $x = \frac{1}{2}$

to $f : y - (\frac{\sqrt{2}}{2} + 2) = (\frac{\sqrt{2}}{2} - 4)(x - \frac{1}{2})$

to $u : y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(x - \frac{1}{2})$

to $v : y - 2 = -4(x - \frac{1}{2})$

The equation of the tangent line to $f$ is the linear combination of the tangent lines to $u$ and to $v$.

3. (a) $y = \sin x - \sqrt{3} \cos x$

$y' = \cos x + \sqrt{3} \sin x$

The tangent line is horizontal when $y' = 0$.

$$0 = \cos x + \sqrt{3} \sin x$$

$$\tan x = -\frac{1}{\sqrt{3}}$$

$$x = -\frac{\pi}{6} + n\pi, \ n = 0, \pm 1, \pm 2, \ldots$$



118-B

(b) $y = 2^x - 2x$

$y' = 2^x \log_e 2 - 2$

This will be perpendicular to $y = 3x + 2$ if $y' = -\frac{1}{3}$.

$$\therefore -\frac{1}{3} = 2^x \log_e 2 - 2$$

$$2^x = \frac{5}{3 \log_e 2}$$

$$x \log_e 2 = \log_e\left(\frac{5}{3 \log_e 2}\right)$$

$$x = \frac{\log_e 5 - \log_e 3 - \log_e(\log_e 2)}{\log_e 2}$$

$$x = \frac{1.61 - 1.79 - \log_e(0.69)}{0.69}$$

$$\approx \frac{1.61 - 1.99 - (-0.37)}{.69}$$

$$\approx -.014$$

(c) If the tangent lines of $y = 5f(x)$ and $y = 7f(x)$ are parallel at $x = a$ then $5f'(a) = 7f'(a)$ or $2f'(a) = 0$ which implies that $f'(a) = 0$. Thus $y = 5f(x)$ and $y = 7f(x)$ both have horizontal tangent lines at $x = a$.

(d) If $u$ and $v$ are differentiable then $f'(x) = u'(x) + 3v'(x)$, and $g'(x) = u'(x) - 11v'(x)$. Since $f'(a) = g'(a)$ then $3v'(a) = -11v'(a)$ or $14v'(a) = 0$ and $v'(a) = 0$ which means that the tangent to $v$ at $(a, v(a))$ is horizontal.

4. If $a$ and $b$ are constants then

$$D(av + bu) = D(av) + D(bu) \qquad \text{by (1)}$$

$$= a\,D(v) + b\,D(u) \qquad \text{by (2)}$$

5. (a) $f: x \to x - \cos x, \quad 0 \le x \le 2\pi$

$f': x \to 1 + \sin x$

$f'': x \to \cos x$

(i) $f$ is increasing in $0 \le x \le 2\pi$

(ii) $f$ is convex in $0 \le x \le \frac{\pi}{2}$

and in $\frac{3\pi}{2} < x \le 2\pi$

$f$ is concave in $\frac{\pi}{2} < x \le \frac{3\pi}{2}$

(iii) No asymptotes.

(b) $f: x \to e^x - 2x, \quad 0 \le x \le 1$

$f': x \to e^x - 2$

$f'': x \to e^x$

(i) $f$ is increasing when $e^x \ge 2$

$0.69 < x \le 1$

$f$ is decreasing when $0 \le x \le 0.69$

(ii) $f$ is everywhere convex.

(iii) No asymptotes in $0 \le x \le 1$.

There is the asymptote $y = -2x$ for $f$ when $x \to -\infty$.

(c) $f: t \to t^2 + \frac{3}{t}, \quad 0 < t$

$f': t \to 2t - \frac{3}{t^2}$

$f'': t \to 2 + \frac{6}{t^3}$

(i) $f$ is increasing when $\sqrt[3]{\frac{3}{2}} \le t$

$f$ is decreasing in $0 < t < \sqrt[3]{\frac{3}{2}}$

(ii) $f$ is convex for $0 < t$

(iii) A vertical asymptote as $t \to 0$.

(d) $f : x \to x^2 - \sqrt{2x}, \quad 0 \le x \le 2$

$$f' : x \to 2x - \frac{\sqrt{2}}{2\sqrt{x}}$$

$$f'' : x \to 2 + \frac{\sqrt{2}}{4x^{3/2}}$$

(i) $f$ is increasing if $\frac{1}{2} < x \le 2$

$f$ is decreasing if $0 \le x \le \frac{1}{2}$

(ii) $f$ is convex in $0 < x \le 2$

(iii) No asymptotes.

6. (a) Let $F'(x) = f(x)$.

Then $$\int_x^b f(x)dx = f(x)\Big]_x^b = F(b) - F(x)$$

$$D\int_x^b f(x)dx = D[F(b) - F(x)]$$

$$= 0 - F'(x) \qquad \text{(from (5) of Section 8-3)}$$

$$= -F'(x)$$

$$= -f(x)$$

(b) Let $F'(x) = f(x)$.

$$D\int_x^0 f(x)dx = -f(x) \qquad \text{(from part (a))}$$

$$D\int_x^0 e^{-t^2}dt = -e^{-x^2}$$

7. The motion of a particle is defined as

$$s(t) = 2\cos t + t^2$$

the velocity as $v(t) = s'(t) = -2\sin t + 2t$ and the acceleration as $a(t) = v'(t) = s''(t) = -2\cos t + 2$. Since $|-2\cos t| \le 2$ then $0 \le -2\cos t + 2 \le 4$ and the acceleration is always nonnegative.

8. Consider the polynomial function $P : x \to a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$.

$$P' : x \to D(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n)$$

$$= D(a_0) + D(a_1x) + D(a_2x^2) + \ldots + D(a_nx^n) \quad \text{by (1)}$$

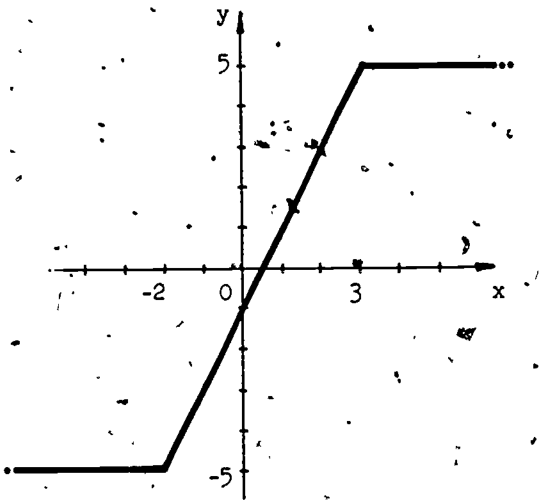
$$= a_0\ D(x^0) + a_1\ D(x) + a_2\ D(x^2) + \ldots a_n\ D(x^n) \quad \text{by (2).}$$

Finally using $Dx^n = nx^{n-1}$.

$$P' : x \to a_0 \cdot 0x^{-1} + a_1 \cdot 1 \cdot x^0 + a_2 \cdot 2x^1 + \ldots + a_n \cdot n \cdot x^{n-1}$$

$$= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + na_nx^{n-1}$$

9. (a)




(b) $$|x + 2| = \begin{cases} x + 2, & \text{if } x \geq -2 \\ -(x + 2), & \text{if } x < -2 \end{cases}$$

$$|3 - x| = \begin{cases} 3 - x, & \text{if } x \leq 3 \\ x - 3, & \text{if } x > 3 \end{cases}$$

$$g(x) = -(x + 2) - (3 - x) = -5, \quad \text{if } x < -2$$
$$g(x) = (x + 2) - (3 - x) = 2x - 1, \quad \text{if } -2 \leq x \leq 3$$
$$g(x) = (x + 2) - (-3 + x) = 5, \quad \text{if } x > 3$$

(c) $f'$ is not defined at $x = -2$ and at $x = 3$.

10. (a) $1 + x + \frac{x^2}{2} \le e^x \le 1 + x + x^2, \quad 0 \le x \le 1.$

Let $f(x) = e^x - (1 + x + \frac{x^2}{2})$

$$f'(x) = e^x - 1 - x$$

The minimum $f$ occurs when $f'(x) = 0$. Since $f'(0) = 0$ we have found at least one minimum.

Let $$g(x) = 1 + x + x^2 - e^x$$

$$g'(x) = 1 + 2x - e^x.$$

Again a minimum occurs when $x = 0$.

(b) Let $$f(x) = v(x) - u(x)$$

Then $$f'(x) = v'(x) - u'(x) \quad \text{, by (1).}$$

Since $v'(x) \ge u'(x)$ it follows that

$$f'(x) \ge 0.$$

Thus, by Theorem 8-2e $f$ is an increasing function. When $a \le x$ then

$$f(a) \le f(x)$$

and $$v(a) - u(a) \le v(x) - u(x).$$

Since $$u(a) \le v(a),$$

$$0 \le v(a) - u(a)$$

implies that $$0 \le v(x) - u(x)$$

and $$u(x) \le v(x) \quad \text{for } a \le x.$$

(c) From part (b) since $u'(a) \le v'(a)$ and $D(u'(x)) \le D(v'(x))$ for $a \le x$ then $u'(x) \le v'(x)$ for $a \le x$.

But we now have exactly the conditions of part (b), thus $u(x) \le v(x)$ for $a \le x$.

11. (a) If $y = u$ and $y = v$ are solutions

$$y'' - 3y' + 6y = 0$$

then $\qquad u'' - 3u' + 6u = 0$

and $\qquad v'' - 3v' + 6v = 0.$

Substitute $\qquad y = 3u + 8v$

$$y' = 3u' + 8v'$$

and $\qquad y'' = 3u'' + 8v''.$

$$(3u'' + 8v'') - 3(3u' + 8v') + 6(3u + 8v)$$
$$= 3(u'' - 3u' + 6u) + 8(v'' - 3v' + 6v)$$
$$3(0) + 8(0) = 0$$

Thus, $y = 3u + 8v$ is also a solution.

(b) If $y = e^x + e^{-x}$ $\qquad$ If $y = e^x - e^{-x}$

$y' = e^x - e^{-x}$ $\qquad$ $y' = e^x + e^{-x}$

$y'' = e^x + e^{-x}$, $\qquad$ $y'' = e^x - e^{-x}$.

In each case $y'' = y$.

If $y = \alpha(e^x + e^{-x}) + \beta(e^x - e^{-x})$ for $\alpha$, $\beta$ constants.

$$y' = \alpha(e^x - e^{-x}) + \beta(e^x + e^{-x})$$

$$y'' = \alpha(e^x + e^{-x}) + \beta(e^x - e^{-x}).$$

Again $y'' = y$.

12. $u(x) = v(x) + ax + b$, for $a$, $b$ constants.

(a) $u'(x) = v'(x) + a$ and $u'(x) - v'(x) = a$.

(b) Since $D(a) = 0$ then $u''(x) = v''(x)$.

(c) If $u'' = v''$ then $u' = v' + a$ by the Constant Difference Theorem. Since $u' = v' + a$ then $u = v + ax + b$ by the Constant Difference Theorem.

Then $u - v = ax + b$, a linear function.

13. If $u$ and $v$ are continuous at $x = a$ then $u(a)$ and $v(a)$ are defined and $\lim_{x \to a} u(x) = u(a)$ and $\lim_{x \to a} v(x) = v(a)$.

Examine
$$f = 2u - 3v$$
$$f(a) = 2u(a) - 3v(a)$$
is defined and
$$\lim_{x \to a} f(x) = \lim_{x \to a} (2u(x) - 3v(x))$$
$$= \lim_{x \to a} 2u(x) - \lim_{x \to a} 3v(x)$$
$$= 2 \lim_{x \to a} u(x) - 3 \lim_{x \to a} v(x)$$
$$= 2u(a) - 3v(a)$$
$$= f(a).$$

Thus $f$ is continuous.

14. The fact that $f$ is differentiable at $x = a$ does not insure that both $u$ and $v$ are also differentiable at $x = a$.

Here are three examples of functions of the form $f = u + v$ such that $f$ is differentiable at $a$ but $u$ and $v$ are not necessarily differentiable at $a$.

(i) $f = |x - a| + (-|x - a|)$

(ii) $f = u + v$, where $u = \begin{cases} 0, & x < a \\ 2x, & a \leq x \end{cases}$

and $v = \begin{cases} 2x, & x < a \\ 0, & a \leq x \end{cases}$

(iii) $f = u + v$, where $u = \begin{cases} x + 1, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$

and $v = \begin{cases} -1, & \text{if } x \text{ is rational} \\ x - 1, & \text{if } x \text{ is irrational} \end{cases}$

1. (a) $a_1 = a^2$, $m_1 = 2a$

$a_2 = a^3$, $m_2 = 3a^2$

(b) $y_1 \cdot y_2 = (a^2 + 2a(x - a))(a^3 + 3a^2(x - a))$

$= a^5 + 5a^4(x - a) + 6a^3(x - a)^2$

If we omit the last term then the expression is $a^5 + 5a^4(x - a)$.
If $u \cdot v = f : x \to x^5$ then $f(a) = a^5$ and $f'(a) = m = 5a^4$.
Thus the tangent line is

$$y = a^5 + 5a^4(x - a),$$

which is the desired result.

2. (a) $Dx(2x - 3) = 4x - 3$

(b) $D(4x - 2)(4 - 2x) = -16x + 20$

(c) $D(x^2 + x + 1)(x^2 - x + 1) = 4x^3 + 2x$

(d) $D\sqrt{x}\,(ax + b)^3 = \dfrac{(ax + b)^3}{2\sqrt{x}} + 3a\sqrt{x}\,(ax + b)^2$

(e) $D(\frac{1}{x} \cdot \sqrt{x}) = -\frac{1}{2}x^{-3/2}$

(f) $D(\frac{1}{x}(5x + 2)) = -\dfrac{2}{x^2}$

(g) $D(xe^x) = e^x + x\,e^x$

$= e^x(1 + x)$

(h) $Dx^{7/2} = \frac{7}{2}x^{5/2}$

(i) $D(3x^4 + \frac{1}{\sqrt{x}}) = 12x^3 - \frac{1}{2}x^{-3/2}$

(j) $D(3x^2(x^2 - 5)) = 12x^3 - 30x$

(k) $D(\sqrt{x}\cos 2x) = -2\sqrt{x}\sin 2x + \dfrac{1}{2\sqrt{x}}\cos 2x$

($\ell$) $D(e^{3x}\sin(x + 1)) = e^{3x}\cos(x + 1) + 3e^{3x}\sin(x + 1)$

$= e^{3x}(\cos(x + 1) + 3\sin(x + 1))$

(m) $D(x^2 \log_e x) = x + 2x \log_e x = x(1 + 2 \log_e x)$

(n) $D((x - 1)^{1/2} e^{-x}) = -(x - 1)^{1/2} e^{-x} + \dfrac{1}{2(x - 1)^{1/2}} e^{-x}$

(o) $D(x\displaystyle\int_0^x e^{-t^2} dt) = xe^{-x^2} + \int_0^x e^{-t^2} dt$

(p) $D(e^x \displaystyle\int_1^x \frac{\sin t}{t} dt) = e^x \frac{\sin x}{x} + e^x \int_1^x \frac{\sin t}{t} dt$

(q) $D(xe^x \sin x) = xe^x \cos x + xe^x \sin x + e^x \sin x$

$= e^x(x \cos x + (x + 1)\sin x)$

(r) $D((\log_e x)(4x^2 + 2x)(\cos 2x))$

$= -2 \log_e x(4x^2 + 2x)\sin 2x + (\log_e x)(\cos 2x)(8x + 2)$

$+ (4x + 2)(\cos 2x)$

(s) $D(2 \sin x \cos x) = -2 \sin^2 x + 2 \cos^2 x$

$= 2(\cos^2 x - \sin^2 x)$

$= 2 \cos 2x$

This was not unexpected since

$$2 \sin x \cos x = \sin 2x$$

and $$D(\sin 2x) = 2 \cos 2x.$$

(t) $D(xe^x \log_e (2x + 1)\sin x) = xe^x \log_e (2x + 1)\cos x + \dfrac{2xe^x}{2x + 1} \sin x$

$+ xe^x \log_e (2x + 1)\sin x$

$+ e^x \log_e (2x + 1)\sin x$

$= e^x(x \log(2x + 1)\cos x + \dfrac{2x}{2x + 1} \sin x$

$+ (x + 1) \log_e (2x + 1) \sin x)$

(u) $D(x^2 2^x) = x^2(\log_e 2)2^x + 2x2^x$

$= x^2 2^x (\frac{2}{x} + \log_e 2)$

(v) $D(x \log_2 (3x + 1)) = \frac{3x}{(\log_e 2)(3x + 1)} + \log_2 (3x + 1)$

$= \frac{1}{\log_e 2} (\frac{3x}{3x + 1} + \log_e (3x + 1))$

(w) $D(x^e e^x) = x^e e^x + e e^x x^{e-1}$

$= x^e e^x (1 + \frac{e}{x})$

3. (a) $D(3x^2 + 5x - 1)^2 = D(3x^2 + 5x - 1)(3x^2 + 5x - 1)$

$= (3x^2 + 5x - 1)(6x + 5) + (6x + 5)(3x^2 + 2x - 1)$

$= 2(6x + 5)(3x^2 + 5x - 1)$

(b) $D(3 - 5x)^3 = D(3 - 5x)(3 - 5x)(3 - 5x)$

$= (3 - 5x)(3 - 5x)(-5) + (3 - 5x)(-5)(3 - 5x)$
$+ (-5)(3 - 5x)(3 - 5x)$

$= -15(3 - 5x)^2$

(c) $D(3 - 5x)^4 = D(3 - 5x)^3(3 - 5x)$

$= (3 - 5x)^3(-5) - 15(3 - 5x)^2(3 - 5x)$

$= -20(3 - 5x)^3$

(d) $D(x(\sqrt{x} - 1)^2) = D(x(\sqrt{x} - 1)(\sqrt{x} - 1))$

$= \frac{x(\sqrt{x} - 1)}{2\sqrt{x}} + \frac{x(\sqrt{x} - 1)}{2\sqrt{x}} + (\sqrt{x} - 1)^2$

$= (\sqrt{x} - 1)(\frac{\sqrt{x}}{2} + \frac{\sqrt{x}}{2} + \sqrt{x} - 1)$

$= (\sqrt{x} - 1)(2\sqrt{x} - 1)$

or $2x - 3\sqrt{x} + 1$

(e) $D(x + \frac{1}{x})^2 = D((x + \frac{1}{x})(x + \frac{1}{x}))$

$= (x + \frac{1}{x})(1 - \frac{1}{x^2}) + (1 - \frac{1}{x^2})(x + \frac{1}{x})$

$= \frac{2}{x}(x^2 + 1) \frac{1}{x^2}(x^2 - 1)$

$= \frac{2}{x^3}(x^4 - 1)$

or $2x - \frac{2}{x^3}$

Alternatively,

$$D(x + \frac{1}{x})^2 = D(x^2 + 2 + \frac{1}{x^2}) = 2x - \frac{2}{x^3}.$$

(f) $D\left(\frac{x^{3/2}}{3} - \frac{x^{1/2}}{2} + x^{-1/2}\right) = \frac{x^{1/2}}{2} - \frac{x^{-1/2}}{4} - \frac{x^{-3/2}}{2}$

(g) $D\left(4\sqrt{x^3} - 2\sqrt{x} + \frac{1}{\sqrt{x}}\right) = 6\sqrt{x} - \frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$.

(h) $D(e^x \sin(1 - 2x)) = -2e^x \cos(1 - 2x) + e^x \sin(1 - 2x)$

$= e^x(-2 \cos(1 - 2x) + \sin(1 - 2x))$

(i) $D(\sqrt{x} \log_e x) = \frac{\sqrt{x}}{x} + \frac{1}{2\sqrt{x}} \log_e x$

$= \frac{1}{\sqrt{x}}(1 + \frac{1}{2} \log_e x)$

or $\frac{1}{\sqrt{x}}(1 + \log_e \sqrt{x})$

(j) $D(x^\pi \pi^x) = x^\pi(\log_e \pi)\pi^x + \pi x^{\pi-1}\pi^x$

$= x^\pi \pi^x(\frac{\pi}{x} + \log_e \pi)$

(k) $D(x^2 \cos x) = -x^2 \sin x + 2x \cos x$

$= x(2 \cos x - x \sin x)$

($\ell$) $D(\sin x \log_e x) = \frac{\sin x}{x} + \cos x \log_e x$

(m) $D\left(\frac{\log_e x}{x}\right) = \frac{1}{x} \cdot \frac{1}{x} + \left(-\frac{1}{x^2}\right)\log_e x$

$= \frac{1}{x^2}(1 - \log_e x)$

4. (a) If $f(x) = [u(x)]^2$

then $f(x) = u(x) \cdot u(x)$

and $f'(x) = u(x)u'(x) + u'(x)u(x)$

$= 2u(x)u'(x)$

(b) If $f(x) = [u(x)]^3$

then $f(x) = u(x)[u(x)]^2$

and $D[u(x)]^3 = u(x)D[u(x)]^2 + u'(x)[u(x)]^2$

$= u(x)[2u(x)(u'(x))] + u'(x)[u(x)]^2$

$= 3[u(x)]^2u'(x)$

(c) $D[u(x)]^4 = D[u(x) \cdot (u(x))^3]$

$= u(x)[3(u(x))^2u'(x)] + u'(x)[u(x)]^3$

$= 4[u(x)]^3u'(x)$

(d) $D[u(x)]^n = n[u(x)]^{n-1}u'(x)$

5\. (a) $y = \sin^2 x$

$y' = 2(\sin x)\cos x$

(b) $y = \cos^3(4x)$

$y' = -12 \cos^2(4x)\sin 4x$

(c) $y = (\log_e x)^2$

$y' = \frac{2}{x} \log_e x$

(d) $y = (e^x)^4$

$y' = 4(e^x)^3 \cdot e^x$

$= 4(e^x)^4$

(e) $y = (x^2 + 1)^2$

$y' = 4x(x^2 + 1)$

(f) $y = \sin^3(2x - 1)$

$y' = 6 \sin^2(2x - 1)\cos(2x - 1)$

(g) $y = (\int_1^x \sin t^2 \, dt)^4$

$y' = 4(\int_1^x \sin t^2 \, dt)^3 \sin x^2$

6. (a) $y = x^2(x^2 + 1)^2$

$$y' = x^2[2(x^2 + 1)2x] + 2x(x^2 + 1)^2$$
$$= 2x(x^2 + 1)(2x^2 + x^2 + 1)$$
$$= 2x(x^2 + 1)(3x^2 + 1)$$

or $6x^5 + 8x^3 + 2x$

(b) $y = (x + 1)^3(x^2 - x + 1)$

$$y' = (x + 1)^3(2x - 1) + 3(x + 1)^2(x^2 - x + 1)$$
$$= (x + 1)^2((x + 1)(2x - 1) + 3(x^2 - x + 1))$$
$$= (x + 1)^2(5x^2 - 2x + 2)$$

(c) $y = (ax^2 + bx + c)(dx^2 + ex + f)$

$$y' = (ax^2 + bx + c)(2dx + e) + (2ax + b)(dx^2 + ex + f)$$
$$= 4adx^3 + 3(ae + bd)x^2 + 2(cd + af + be)x + (ce + bf)$$

(d) $y = (\cos^2 x)\sin 2x$

$$y' = 2\cos^2 x \cos 2x - 2\cos x \sin x \sin 2x$$
$$= 2\cos x(\cos x \cos 2x - \sin x \sin 2x)$$
$$= 2\cos x \cos 3x$$

(e) $y = e^x \sin^2(ax + b)$

$$y' = 2ae^x \sin(ax + b)\cos(ax + b) + e^x \sin^2(ax + b)$$
$$= e^x \sin(ax + b)(2a\cos(ax + b) + \sin(ax + b))$$

(f) $y = \left(x\int_0^x e^{t^2}\,dt\right)^2 = x^2\left(\int_0^x e^{t^2}\,dt\right)^2$

$$y' = 2x^2e^{x^2}\left(\int_0^x e^{t^2}\,dt\right)^1 + 2x\left(\int_0^x e^{t^2}\,dt\right)^2$$
$$= 2x\int_0^x e^{t^2}\,dt\left[xe^{x^2} + \int_0^x e^{t^2}\,dt\right]$$

(g) $y = x^3[\log_e(x+1)]^3$

$$y' = \frac{3x^3}{x+1}[\log_e(x+1)]^2 + 3x^2[\log_e(x+1)]^3$$

$$= 3x^2[\log_e(x+1)]^2[\frac{x}{x+1} + \log_e(x+1)]$$

7. (a) $y = x \log_e x$ , $x > 0$

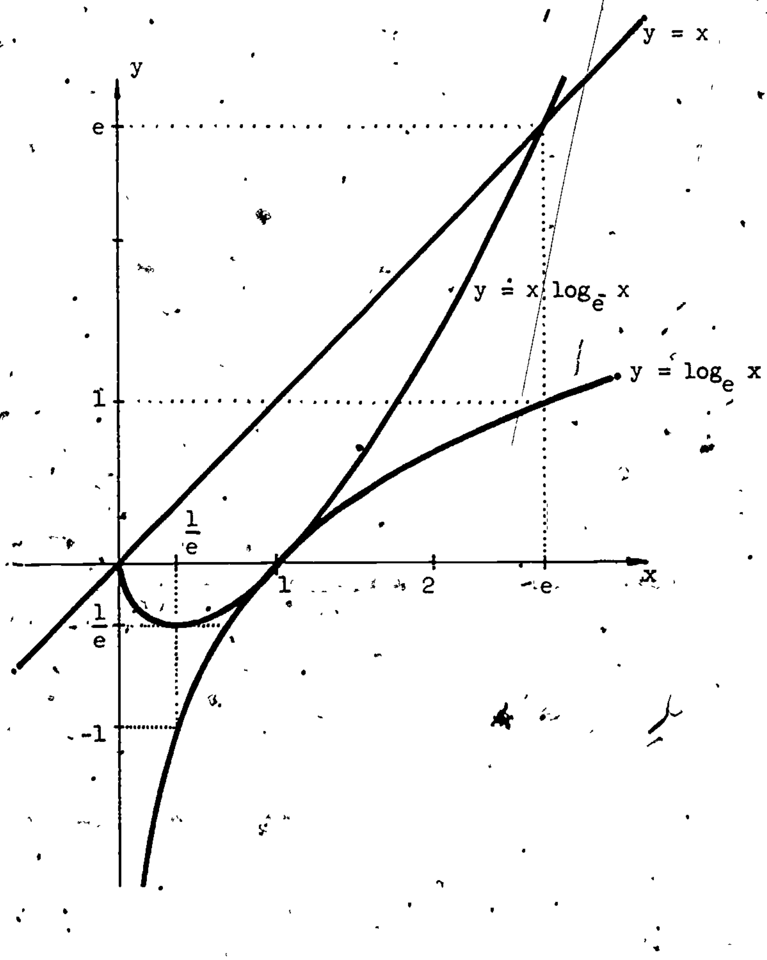
$y' = 1 + \log_e x$

$y'' = \frac{1}{x}$

The graph increases if $x \geq \frac{1}{e}$

The graph decreases if $0 < x < \frac{1}{e}$

The graph is convex if $0 < x$



(b) $y = \frac{1}{x}\log_e x, \quad x > 0$

$$y' = \frac{1}{x}\cdot\frac{1}{x} - \frac{1}{x^2}\log_e x$$

$$= \frac{1}{x^2}(1 - \log_e x)$$

$y$ is increasing when $\frac{1}{x^2}(1 - \log_e x) > 0$ or when $x \leq e$.

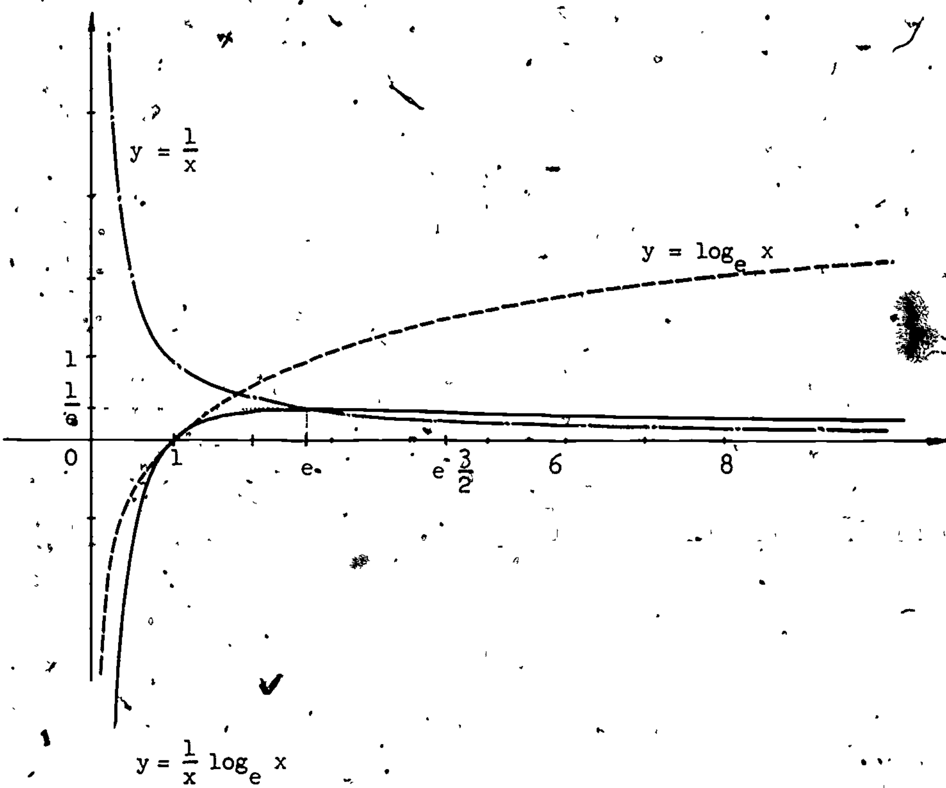
$y$ is decreasing when $x > e$.

$$y'' = \frac{1}{x^2}\left(-\frac{1}{x}\right) + \left(-\frac{2}{x^3}\right)(1 - \log_e x)$$

$$= \frac{1}{x^3}(2\log_e x - 3)$$

$y$ is convex if $\frac{1}{x^2}(2\log_e x - 3) \geq 0$ or when $x \geq e^{3/2} \approx 4.48$.

$y$ is concave if $0 < x < e^{3/2}$



(c) $y = \sin 3x, \quad 0 \le x \le 2\pi$

$y' = 3 \sin^2 x \cos x$

$y'' = 3(-\sin^3 x + 2 \sin x \cos^2 x)$

We see that $y'$ depends solely upon $\cos x$ for its being positive or negative.

Thus $y$ is increasing when $\cos x \ge 0$, that is when $0 \le x \le \frac{\pi}{2}$ or $\frac{3\pi}{2} \le x \le 2\pi$ and $y$ is decreasing when $\frac{\pi}{2} \le x < \frac{3\pi}{2}$. Analyzing $y''$ is more involved.

$$y'' = 3 \sin^3 x(2 \cot^2 x - 1).$$

We see that $y'' \ge 0$ if

(i) both $\sin^3 x \ge 0$

and $2 \cot^2 x - 1 \ge 0$ or

(ii) both $\sin^3 x < 0$

and $2 \cot^2 x - 1 < 0.$

Case (i): $\sin^3 x \ge 0$ when $0 \le x < \pi$

$2 \cot^2 x - 1 \ge 0$

$\cot^2 x \ge \frac{1}{2}$

$$|\cot x| \ge .707, \quad \cot .955 \approx .707$$

Thus

$$0 \le x < .955 \text{ or } \pi - .955 \le x < \pi + .955 \text{ or } \pi - .955 \le x \le 2\pi.$$

Combining both conditions of case (i) $y'' \ge 0$ when

$$0 \le x < .955 \text{ or } \pi - .955 \le x < \pi.$$

Case (ii): $\sin^3 x \le 0$ when $\pi \le x \le 2\pi$

$2 \cot^2 x - 1 \le 0$ when $.955 \le x < \pi - .955$ or $\pi + .955 \le x < 2\pi - .955$

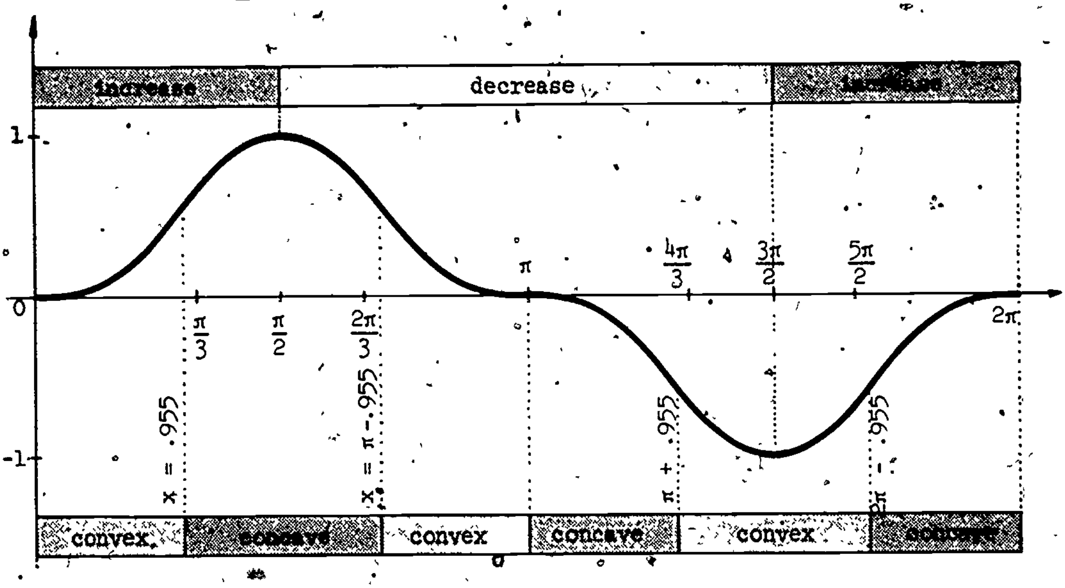
Combining both conditions of case (ii) $y'' \ge 0$ when

$$\pi + .955 \le x < 2\pi - .955.$$

Thus $y$ is convex in the following domain $0 \le x < .955$, $\pi - .955 \le x < \pi$, or $\pi + .955 \le x < 2\pi - .955$.

We can assume that $y$ is concave in the complement of the domain for which $y'' \geq 0$, namely

$$.955 \leq x < \pi - .955, \quad \pi \leq x < \pi + .955, \quad \text{or} \quad 2\pi - .955 \leq x \leq 2\pi.$$




(d) $y = x^2 \log_e x, \quad x > 0$

$$y' = x + 2x \log_e x$$

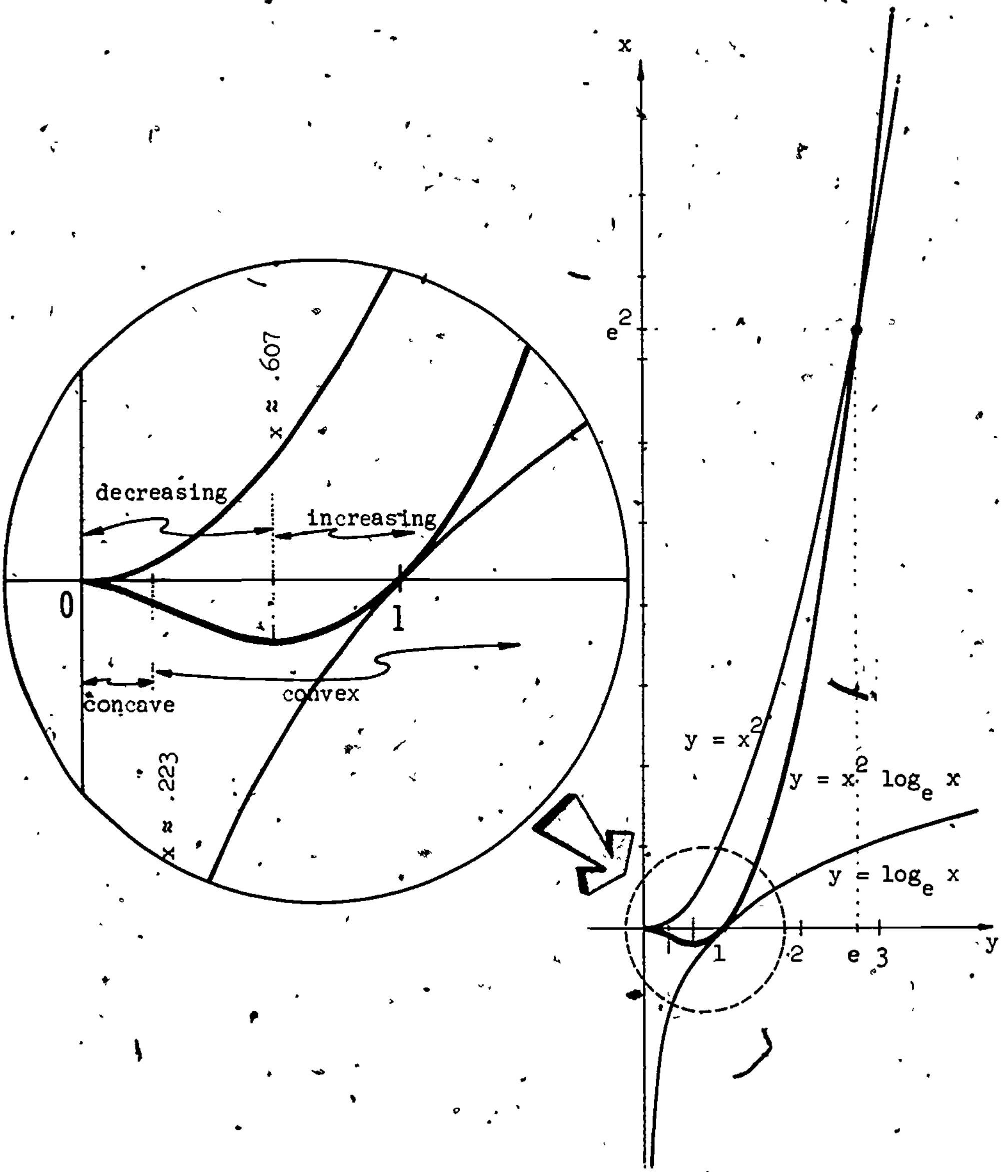
$$= x(1 + 2 \log_e x)$$

$$y'' = 2 + (1 + 2 \log_e x)$$

$$= 3 + 2 \log_e x$$

Since $x > 0$ then $y' \geq 0$ when $1 + 2 \log_e x \geq 0$ or when $x \geq e^{-1/2} \approx .607$. Thus $y$ increases when $x \geq e^{-1/2}$ and decreases if $0 < x < e^{-1/2}$.

The function is convex when $3 + 2 \log_e x \geq 0$ or when $x \geq e^{-3/2} \approx .223$ and concave for $x \leq e^{-3/2}$.




8. (a) $x \to \sqrt{x}\, e^x, \quad x > 0$

$$D(\sqrt{x}\, e^x) = \sqrt{x}\, e^x + \frac{e^x}{2\sqrt{x}}$$

$$= \sqrt{x}\, e^x\left(1 + \frac{1}{2x}\right)$$

Since $x > 0$ then $\sqrt{x} > 0$, $e^x > 0$ and $\left(1 + \frac{1}{2x}\right) > 0$.

Thus $D(\sqrt{x}\, e^x) > 0$ and the function is increasing.

(b) $x \to \frac{e^x}{x}, \ x \geq 1$

$$D\left(\frac{e^x}{x}\right) = \frac{e^x}{x} - \frac{e^x}{x^2}$$

$$= \frac{e^x}{x^2}(x - 1)$$

If $x \geq 1$ then $e^x > 0$, $x^2 > 0$ and $(x - 1) \geq 0$.

Thus $D\left(\frac{e^x}{x}\right) \geq 0$ and the function is increasing.

(c) $x \to \frac{e^x}{x^\alpha}, \ x \geq \alpha > 0$

$$D\left(\frac{e^x}{x^\alpha}\right) = \frac{e^x}{x^\alpha} - \alpha \frac{e^x}{x^{\alpha+1}}$$

$$= \frac{e^x}{x^{\alpha+1}}(x - \alpha).$$

If $x \geq \alpha > 0$ then $e^x > 0$, $x^{\alpha+1} > 0$ and $(x - \alpha) \geq 0$.

Thus $D\left(\frac{e^x}{x^\alpha}\right) \geq 0$ and the function is increasing.

(d) $x \to x \sin x, \ 0 \leq x \leq \frac{\pi}{2}$

$D(x \sin x) = x \cos x + \sin x$

If $0 \leq x \leq \frac{\pi}{2}$ then $x > 0$, $\sin x \geq 0$ and $\cos x \geq 0$.

Thus $D(x \sin x) \geq 0$ and the function is increasing.

9. If $f(x) = (x - a)^2 g(x)$, $g(a) \neq 0$ and $g$ is differentiable then

$$f'(x) = (x - a)^2 g'(x) + 2(x - a)g(x).$$

Since $(x - a)$ is a factor of each term then $f'(a) = 0$.

10. Let $D : x \to a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial function with the zero $a$ of multiplicity at least 3. Then by the Factor Theorem $P(x) = (x - a)^3Q(x)$ where $Q(x)$ is a polynomial of degree $(n - 3)$.

$$P'(x) = (x - a)^3Q'(x) + 3(x - a)^2Q(x)$$

$$P''(x) = (x - a)^3Q''(x) + 3(x - a)^2Q'(x) + 3(x - a)^2Q'(x) + 6(x - a)Q(x)$$

The expression for $P''(x)$ has a factor of $(x - a)$ in each term. Thus, $P''(a) = 0$ whenever $a$ is a zero of the polynomial of multiplicity greater than two.

The converse is not true. A counter example is called for. Assume $a \neq 0$.

Suppose that $\quad P''(x) = x - a$, and $P''(a) = 0$.

Further $\quad P'(x) = \frac{x^2}{2} - ax$,

and $\quad P(x) = \frac{x^3}{6} - \frac{ax^2}{2}$.

But $\quad P(a) \neq 0$.

11. (a) If $\quad y = e^{ax} \cos bx$

then $\quad y' = -be^{ax} \sin bx + ae^{ax} \cos bx$

and $\quad y'' = -b^2 e^{ax} \cos bx - ab\, e^{ax} \sin bx - ab\, e^{ax} \sin bx + a^2 e^{ax} \cos bx$

$$= e^{ax}(a^2 \cos bx - 2ab \sin bx - b^2 \cos bx).$$

Substituting into $y'' - 2ay + (a^2 + b^2)y$ we get

$$e^{ax}(a^2 \cos bx - 2ab \sin bx - b^2 \cos bx) - e^{ax}(-2ab \sin bx + 2a^2 \cos bx) + e^{ax}(a^2 \cos bx + b^2 \cos bx).$$

Simplifying yields

$$e^{ax}((a^2 - 2a^2 + a^2 - b^2 + b^2)\cos bx + (-2ab + 2ab)\sin bx)$$

or

$$e^{ax}(0 \cdot \cos bx + 0 \cdot \sin bx)$$

which is clearly zero.

(b) If $\qquad y = x^2e^x + 2xe^x = e^x(x^2 + 2x)$

then $\qquad y' = e^x(2x + 2) + e^x(x^2 + 2x),$

$$= e^x(x^2 + 4x + 2)$$

$$y'' = e^x(2x + 4) + e^x(x^2 + 4x + 2)$$

$$= e^x(x^2 + 6x + 6)$$

and $\qquad y''' = e^x(2x + 6) + e^x(x^2 + 6x + 6)$

$$= e^x(x^2 + 8x + 12)$$

Substituting into $y''' - 3y'' + 3y' - y$ gives

$$e^x(x^2 + 8x + 12) - e^x(3x^2 + 18x + 18) + e^x(3x^2 + 12x + 6) - e^x(x^2 + 2x).$$

Simplifying yields

$$e^x((1 - 3 + 3 - 1)x^2 + (8 - 18 + 12 - 2)x + (12 - 18 + 6))$$

or $\qquad e^x(0 \cdot x^2 + 0 \cdot x + 0)$

which is clearly zero.

12. (a) $(uv)' = uv' + vu'$

$(uv)'' = uv'' + u'v' + u'v' + u''v$

$\qquad = uv'' + 2u'v' + u''v.$

(b) $f : x \rightarrow x^2 \cos x$

Let $u = x^2$ and $v = \cos x$. Then $u' = 2x$, $u'' = 2$, $v' = -\sin x$ and $v'' = -\cos x$.

Thus from (a)

$$f''(x) = -x^2 \cos x - 4x \sin x + 2 \cos x.$$

(c) $(uv)''' = uv''' + u'v'' + 2u'v'' + 2u''v' + u'''v + u'' + v'$

$\qquad = uv''' + 3u'v'' + 3u''v' + u'''v.$

(d) $$(uv)^n = uv^{(n)} + nu'v^{(n-1)} + \frac{n \cdot (n - 1)}{2} u''v^{(n-2)} + \ldots$$

$$+ \frac{n!}{m!(n - m)!} u^m v^{(n-m)} + \ldots$$

$$+ nu^{(n-1)}v' + u^{(n)}v \qquad \text{for } m \leq n.$$

The coefficients are the coefficients of the binomial expansion.

## Solutions Exercises 8-7

1. (a) $x \to \sqrt{1 - x^2}$

Let $f(u) = \sqrt{u}$

and $u(x) = 1 - x^2$.

Or let $f(u) = \sqrt{1 - u}$

and $u(x) = x^2$.

Then $f(u(x)) = \sqrt{1 - x^2}$.

(b) $x \to e^{x^2}$

Let $f(u) = e^u$

and $u(x) = x^2$.

Then $f(u(x)) = e^{x^2}$.

(c) $x \to \cos(x^3 - 3x)$

Let $f(u) = \cos u$

and $u(x) = x^3 - 3x$.

Then $f(u(x)) = \cos(x^3 - 3x)$.

(d) $x \to \dfrac{1}{1 + x^2}$

Let $f(u) = \dfrac{1}{u}$

and $u(x) = 1 + x^2$.

Or let $f(u) = \dfrac{1}{1 + u}$

and $u(x) = x^2$.

Then $f(u(x)) = \dfrac{1}{1 + x^2}$.

(e) $x \to \log_e \sqrt{x^2 + 1}$

Let $f(u) = \log_e u$

and $u(x) = \sqrt{x^2 + 1}$.

Or let $f(u) = \log_e \sqrt{u + 1}$

and $u(x) = x^2$.

Or let $f(u) = \frac{1}{2}\log_e u$

and $u(x) = x^2 + 1$.

Or let $f(u) = \frac{1}{2}\log(u + 1)$

and $u(x) = x^2$.

Then $f(u(x)) = \frac{1}{2}\log_e (x^2 + 1)$

$= \log_e \sqrt{x^2 + 1}$

(f) $x \to (2 - 3x^2)^{100}$

Let $f(u) = (2 - u)^{100}$

and $u(x) = 3x^2$.

Or let $f(u) = u^{100}$.

and $u(x) = 2 - 3x^2$.

Then $f(u(x)) = (2 - 3x^2)^{100}$.

(g) $x \to (2x^2 - 2x + 1)^{-1/2}$

Let $f(u) = u^{-1/2}$

and $u(x) = 2x^2 - 2x + 1$.

Then $f(u(x)) = (2x^2 - 2x + 1)^{-1/2}$.

(h) $x \to \log_e(\sin x)^2$

Let $f(u) = \log_e u$

and $u(x) = \sin^2 x$.

Or let $f(u) = \log_e u^2$

and $u(x) = \sin x$.

Or let $f(u) = 2\log_e u$

and $u(x) = \sin x$

Then $f(u(x)) = 2\log_e \sin x$

$= \log_e(\sin x)^2$.

(i) $x \to e^{\cos^2 x}$

Let $f(u) = e^u$

and $u(x) = \cos^2 x$.

Or let $f(u) = e^{u^2}$

and $u(x) = \cos x$.

Then $f(u(x)) = e^{(\cos x)^2}$

(j) $x \to 3e^{2 \sin x}$

Let $f(u) = 3u$

and $u(x) = e^{2 \sin x}$.

Or let $f(u) = 3e^u$

and $u(x) = 2 \sin x$.

Or let $f(u) = 3e^{2u}$

and $u(x) = \sin x$

Then $f(u(x)) = 3e^{2 \sin x}$.

(k) $x \to 2^{(x+1)^2}$

Let $f(u) = 2^u$

and $u(x) = (x + 1)^2$.

Or let $f(u) = 2^{u^2}$

and $u(x) = x + 1$.

Then $f(u(x)) = 2^{(x+1)^2}$.

2. (a) $x \to \log_e |8x^2 + 5x + 2|$

Let $f(u) = \log_e u$

$u(v) = |v|$

and $v(x) = 8x^2 + 5x + 2$.

Then $f(u(v(x))) = \log_e |8x^2 + 5x + 2|$.

(b) $x \to \sqrt{1 + \cos x}$

Let $f(u) = \sqrt{u}$

$u(v) = 1 + v$

and $v(x) = \cos x.$

Then $f(u(v(x))) = \sqrt{1 + \cos x}.$

(c) $x \to \cos(\sin(\cos x))$

Let $f(u) = \cos u$

$u(v) = \sin v$

and $v(x) = \cos x.$

Then $f(u(v(x))) = \cos(\sin(\cos x)).$

(d) $x \to (x + 1)^{3/5}$

Let $f(u) = (u)^{1/5}$

$u(v) = v^3$

and $v(x) = x + 1.$

Or let $f(u) = u^3$

$u(v) = v^{1/5}$

and $v(x) = x + 1.$

Then $f(u(v(x))) = (x + 1)^{3/5}.$

(e) $x \to \sqrt{1 - (\log_e x)^2}$

Let $f(u) = \sqrt{u}$

$u(v) = 1 - v$

$v(q) = q^2$

and $q(x) = \log_e x.$

Then $f(u(v(q(x)))) = \sqrt{1 - (\log_e x)^2}.$

(f) $x \to \dfrac{1}{1 + e^{2x}}$

Let $f(u) = \dfrac{1}{u}$

$u(v) = 1 + v$

$v(q) = e^q$

$q(x) = 2x$

Then $f(u(v(q(x)))) = \dfrac{1}{1 + e^{2x}}$

3. If $x \to |x|$ then let $f(u) = \sqrt{u}$ and $u(x) = x^2$. Thus $f(u(x)) = \sqrt{x^2}$ which is another way of saying $|x|$.

4. (a) Let $f(u) = au + b$ and $u(x) = px + q$ be linear functions.

Then
$$f(u(x)) = a(px + q) + b$$
$$= apx + aq + b$$

which is a linear function.

(b) Let $f(u) = au^2 + bu + c$

and $u(x) = rx^2 + sx + t$

be quadratic functions.

Then

$$f(u(x)) = a(rx^2 + sx + t)^2 + b(rx^2 + sx + t) + c$$
$$= a(r^2x^4 + s^2x^2 + t^2 + 2rsx^3 + 2rtx^2 + 2stx)$$
$$+ b(rx^2 + sx + t) + c$$
$$= ar^2x^4 + 2rsx^3 + (as^2 + wrt + br)x^2$$
$$+ (2st + bs)x + (at^2 + bt + c)$$

which is a fourth degree polynomial.

(c) The composite of two polynomial functions is again a polynomial function. The degree of the composite polynomial function will be equal to the product of the degrees of the two contributing functions.

5. (a) If $u : x \to x$ and $f : x \to u(u(x))$ then

$$f(3) = u(u(3))$$
$$= u(3)$$
$$= 3.$$

(b) If $u : x \to \frac{1}{x}$ then $f : x \to u(u(x))$. Thus

$f(x) = u(u(x)) = u(\frac{1}{x}) = x$ and $f(x) = x$.

6. (a) If $u : x \to x^a$ and $v : x \to x^b$ then

$$u(v(x)) = u(x^b)$$
$$= (x^b)^a$$
$$= x^{ab}$$

and
$$v(u(x)) = v(x^a)$$
$$= (x^a)^b$$
$$= x^{ab}.$$

Thus
$$u(v(x)) = v(u(x)).$$

(b) If $u : x \to \cos x$ and $v : x \to \sin x$ then

$$u(v(x)) = \cos(\sin x)$$

and
$$v(u(x)) = \sin(\cos x).$$

A counter example will establish the fact that $u(v(x)) \neq v(u(x))$.

Let
$$x = \frac{\pi}{2},$$

then
$$u(v(x)) = \cos(\sin \frac{\pi}{2})$$
$$= \cos 1$$
$$\approx .5403.$$

But
$$v(u(x)) = \sin(\cos \frac{\pi}{2})$$
$$= \sin 0$$
$$= 0$$

$$0 \neq .5403$$

Therefore
$$u(v(x)) \neq v(u(x)).$$

(c) Let $u : x \to a^x$ and $v : x \to b^x$

$$u(v(x)) = a^{(b^x)}$$

and
$$v(u(x)) = b^{(a^x)}.$$

In general $u(v(x)) \neq v(u(x))$ unless $a = b$. The counter example which gives the quickest results is to let $x = 0$.

(d) Let $u : x \to e^x$ and $v : x \to \log_e x$ then

$$u(v(x)) = e^{\log_e x} = x$$

and $$v(u(x)) = \log_e e^x = x.$$

In this case $u(v(x)) = v(u(x))$.

This is no surprise since $\log_e x$ was introduced as the inverse function of $e^x$.

7. (a) $x \to \int_{-2}^{x^2} t^{2/3}\, dt$

Let $$f(u) = \int_{-2}^{u} t^{2/3}\, dt$$

and $$u(x) = x^2,$$

then $$f(u(x)) = \int_{-2}^{x^2} t^{2/3}\, dt.$$

(b) $x \to \int_{\sin x}^{1} e^t\, dt$

Let $$f(u) = \int_{u}^{1} e^t\, dt$$

and $$u(x) = \sin x.$$

Then $$f(u(x)) = \int_{\sin x}^{1} e^t\, dt.$$

(c) $x \to \int_{0}^{x^2} e^{-t^2}\, dt$

Let $$f(u) = \int_{0}^{u} e^{-t^2}\, dt$$

and $$u(x) = x^2.$$

Then $$f(u(x)) = \int_{0}^{x^2} e^{-t^2}\, dt.$$

8. $x \to \sqrt{1 - (\log_e x)^2}$

First, $\log_e x$ is only defined when $x > 0$. Secondly, $\sqrt{1 - (\log_e x)^2}$ is only defined for real numbers when

$$1 - (\log_e x)^2 \geq 0.$$

Thus

$$-(\log_e x)^2 \geq -1$$

$$(\log_e x)^2 \leq 1$$

$$|\log_e x| \leq 1$$

$$-1 \leq \log_e x \leq 1.$$

$$\frac{1}{e} \leq x \leq e$$

The domain of the function is the interval $\frac{1}{e} \leq x \leq e$.

## Solutions Exercises 8-8

1. (a) $x \to \sqrt{1 - x^2}$

Let $$f(u) = u^{1/2}$$

and $$u(x) = 1 - x^2.$$

Then $$f'(x) = f'(u) \cdot u'(x) = \frac{1}{2\sqrt{1 - x^2}} \cdot -2x$$

$$= \frac{-x}{\sqrt{1 - x^2}}$$

(b) $x \to e^{x^2}$

$$f(u) = e^u$$

$$u(x) = x^2$$

$$f'(x) = e^{x^2} \cdot 2x$$

$$= 2x\, e^{x^2}$$

(c) $x \to \cos (x^3 - 3x)$

$$f(u) = \cos u$$

$$u(x) = x^3 - 3x$$

$$f'(x) = -\sin(x^3 - 3x) \cdot (3x^2 - 3)$$

$$= (3 - 3x^2)\sin(x^3 - 3x)$$

(d) $x \to \frac{1}{1 + x^2}$

$$f(u) = \frac{1}{u}$$

$$u(x) = 1 + x^2$$

$$f'(x) = -\frac{1}{(1 + x^2)^2} \cdot (2x)$$

$$= \frac{-2x}{(1 + x^2)^2}$$

(e) $x \to \log_e \sqrt{x^2 + 1}$

$$f(u) = \log u$$

$$u(v) = \sqrt{v}$$

$$v(x) = x^2 + 1$$

$$f'(x) = \frac{1}{\sqrt{x^2 + 1}} \cdot \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x$$

$$= \frac{x}{x^2 + 1}$$

(f) $x \to (2 - 3x^2)^{100}$

$$f(u) = u^{100}$$

$$u(x) = 2 - 3x^2$$

$$f'(x) = 100(2 - 3x^2)^{99} \cdot (-6x)$$

$$= -600x(2 - 3x^2)^{99}$$

(g) $x \to (2x^2 - 2x + 1)^{-1/2}$

$$f(u) = u^{-1/2}$$

$$u(x) = 2x^2 - 2x + 1$$

$$f'(x) = -\frac{1}{2}(2x^2 - 2x + 1)^{-3/2} \cdot (4x - 2)$$

$$= (1 - 2x)(2x^2 - 2x + 1)^{-3/2}$$

(h) $x \to \log_e (\sin x)^2$

$$f(u) = \log u$$

$$u(v) = v^2$$

$$v(x) = \sin x$$

$$f'(x) = \frac{1}{\sin^2 x} \cdot 2 \sin x \cdot \cos x$$

$$= 2 \cot x$$

(i) $x \to e^{\cos^2 x}$

$$f(u) = e^u$$

$$u(v) = v^2$$

$$v(x) = \cos x$$

$$f'(x) = e^{\cos^2 x} \cdot 2 \cos x \cdot (-\sin x)$$

$$= -2e^{\cos^2 x}(\cos x \sin x)$$

or $$-e^{\cos^2 x} (\sin 2x)$$

(j) $x \to 3e^{2 \sin x}$

$$f(u) = 3e^{2u}$$

$$u(x) = \sin x$$

$$f'(x) = 6e^{2 \sin x} (\cos x)$$

(k) $x \to 2^{(x+1)^2}$

$$f(u) = 2^u$$

$$u(v) = v^2$$

$$v(x) = x + 1$$

$$f'(x) = (\log_e 2) \cdot (2^{(x+1)^2}) \cdot 2(x + 1) \cdot (1)$$

$$= 2(x + 1)(\log_e 2)(2^{(x+1)^2})$$

2. (a) $x \to \sqrt{1 + \cos x}$

$$f(u) = \sqrt{u}$$

$$u(x) = 1 + \cos x$$

$$f'(x) = \frac{1}{2\sqrt{1 + \cos x}} \cdot (-\sin x)$$

$$= \frac{-\sin x}{2\sqrt{1 + \cos x}}$$

(b) $x \to \sqrt{1 - (\log_e x)^2}$

$$f(u) = \sqrt{u}$$

$$u(v) = 1 - v^2$$

$$v(x) = \log_e x$$

$$f'(x) = \frac{1}{2\sqrt{1 - (\log_e x)^2}} \cdot (-2 \log_e x) \cdot \frac{1}{x}$$

$$= \frac{-\log_e x}{x\sqrt{1 - (\log_e x)^2}}$$

(c) $x \to \frac{1}{1 + e^{2x}}$

$$f(u) = \frac{1}{u}$$

$$u(x) = 1 + e^{2x}$$

$$f'(x) = \frac{-1}{(1 + e^{2x})^2} \cdot 2e^{2x}$$

$$= \frac{-2e^{2x}}{(1 + e^{2x})^2}$$

(d) $x \to \cos(\sin(\cos x))$

$$f(u) = \cos u$$

$$u(v) = \sin v$$

$$v(x) = \cos x$$

$$f'(x) = [-\sin(\sin(\cos x))](\cos(\cos x))(-\sin x)$$

3. (a) $x \to (x^2 + 1)^{1/2} + (x^2 + 1)^{-1/2}$

$$f'(x) = \frac{1}{2(x^2 + 1)^{1/2}} \cdot 2x + \frac{-1}{2(x^2 + 1)^{3/2}} \cdot 2x$$

$$= \frac{x(x^2 + 1) - x}{(x^2 + 1)^{3/2}}$$

$$= \frac{x^3}{(x^2 + 1)^{3/2}}$$

(b) $x \to (x^2 - a^2)^{1/2} \cdot (x^2 + a^2)^{-1/2}$

$$f'(x) = (x^2 - a^2)^{1/2}(-\tfrac{1}{2})(x^2 + a^2)^{-3/2}(2x) + (x^2 + a^2)^{-1/2}(\tfrac{1}{2})$$

$$(x^2 - a^2)^{-1/2}(2x)$$

$$= (x^2 - a^2)^{-1/2}(x^2 + a^2)^{-3/2}(-x(x^2 - a^2) + x(x^2 + a^2))$$

$$= 2a^2 x(x^2 - a^2)^{-1/2}(x^2 + a^2)^{-3/2}$$

or $$\frac{2a^2x}{(x^2 - a^2)^{1/2}(x^2 + a^2)^{3/2}}$$

(c) $x \to x(2x^2 + 2x + 1)^{-1/2}$

$$f'(x) = x(-\tfrac{1}{2})(2x^2 + 2x + 1)^{-3/2}(4x + 2) + (2x^2 + 2x + 1)^{-1/2}$$

$$= (2x^2 + 2x + 1)^{-3/2}(-(2x^2 + x) + (2x^2 + 2x + 1))$$

$$= (x + 1)(2x^2 + 2x + 1)^{-3/2}$$

(d) $x \to x^2 \sqrt{\sin x}$

$$f'(x) = x^2 \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x + \sqrt{\sin x}\,(2x)$$

$$= \frac{x}{2}\sqrt{\sin x}\,(x \cot x + 4)$$

(e) $x \to \sin^2(e^x)$

$$f'(x) = 2 \sin e^x \cdot \cos e^x \cdot e^x$$

$$= 2e^x \sin e^x \cos e^x$$

or $\quad e^x \sin 2e^x$

(f) $x \to e^{x \sin x}$

$$f'(x) = e^{x \sin x}(x \cos x + \sin x)$$

(g) $x \to \log_e(\sqrt{x} \cos x)$

$$f'(x) = \frac{1}{\sqrt{x} \cos x} \cdot \left(\sqrt{x}(-\sin x) + \frac{1}{2\sqrt{x}} \cdot \cos x\right)$$

$$= -\tan x + \frac{1}{2x}$$

(h) $x \to e^{\log_e x + \cos x}$

$$f'(x) = \left(\frac{1}{x} - \sin x\right)e^{\log_e x + \cos x}$$

(i) $x \to \sin x \cos x \log_e \sqrt{x}$

$$f'(x) = \sin x \cos x\left(\frac{1}{2x}\right) + \sin x(-\sin x)\log_e\sqrt{x} + \cos x \cos x \log_e\sqrt{x}$$

$$= \frac{1}{2x} \sin x \cos x + \log_e\sqrt{x}\,(\cos^2 x - \sin^2 x)$$

or $\quad \frac{1}{4x}(\sin 2x) + \log_e\sqrt{x}\,(\cos 2x)$

Alternate method

$$x \to \frac{1}{2} \sin 2x \log_e \sqrt{x}$$

$$f'(x) = \frac{1}{2} \sin 2x[\frac{1}{2x}] + (\frac{2}{2})\log_e\sqrt{x}\ (\cos 2x)$$

$$= \frac{1}{4x} (\sin 2x) + (\log_e \sqrt{x})(\cos 2x)$$

(j) $x \to \cos^2(\log_e x) + \sin^2(\log_e x)$

$$f'(x) = 0$$

4. (a) $f(x) = \int_a^{g(x)} h(t)dt$

Let $u = g(x)$

$$f(u) = \int_a^u h(t)dt$$

then $f'(u) = h(u)$

and $f'(x) = f'(u) \cdot g'(x)$

$$= h(g(x))g'(x)$$

(b) $F(x) = \int_{x^2}^b f = -\int_b^{x^2} f$

Let $x^2 = u.$

Then $F'(u) = -f(u)$

and $u'(x) = 2x.$

Thus $F'(x) = -f(x^2)2x$

$$= -2x\ f(x^2)$$

(c) $f(x) = \int_{-\pi}^{x^2} \sin t \, dt = -\cos t \Big|_{-\pi}^{x^2}$

$$= (-\cos x^2) - (-\cos(-\pi))$$

$$= -\cos x^2 - 1$$

$$f'(x) = -(-\sin x^2)(2x)$$

$$= 2x \sin x^2$$

If we allow $x^2 = u$ then the first result is the same as

$$f'(u) \cdot u'(x) = \sin(x^2)(2x).$$

5. (a) $x \to \int_{-2}^{x^2} t^{2/3} \, dt$

$$f'(x) = (x^2)^{2/3} \cdot 2x$$

$$= 2x^{7/3}$$

(b) $x \to \int_{\sin x}^{1} e^t \, dt$

$$f'(x) = -e^{\sin x} \cdot (\cos x)$$

(c) $x \to \int_0^{x^2} e^{-t^2} \, dt$

$$f'(x) = e^{-x^4} \cdot 2x$$

$$= 2xe^{-x^4}$$

6. (a) $f : x \to x^x, \quad x > 0$

$f : x \to e^{x \log_e x}$

$$f'(x) = e^{x \log_e x} \cdot (x \cdot \frac{1}{x} + \log_e x)$$

$$= x^x(1 + \log_e x)$$

(b) Minimum value of $f$ occurs when $f' = 0$.

Since $x > 0$ then $x^x > 0$.

Thus $f' = 0$ when $1 + \log_e x = 0$

$$1 + \log_e x = 0$$

$$\log_e x = -1$$

$$x = e^{-1}$$

The minimum value of $f$ is

$$f(e^{-1}) = (e^{-1})^{e^{-1}}$$

$$= (\tfrac{1}{e})^{1/e} \approx (.3679)^{(.3679)} \approx .711$$

(c) $f''(x) = x^x(\frac{1}{x}) + x^x(1 + \log_e x)^2$

$$= x^x(\frac{1}{x} + (1 + \log_e x)^2)$$

By inspection since $x > 0$ then $f'' > 0$ and $f$ is convex.

7. $f : x \mapsto \dfrac{x}{x^2 - 1}$

$$f'(x) = x \cdot \frac{-1}{(x^2 - 1)^2} \cdot 2x + \frac{1}{x^2 - 1}$$

$$= \frac{-2x^2 + (x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{-x^2 - 1}{(x^2 - 1)^2} = \frac{-(x^2 + 1)}{(x^2 - 1)^2}$$

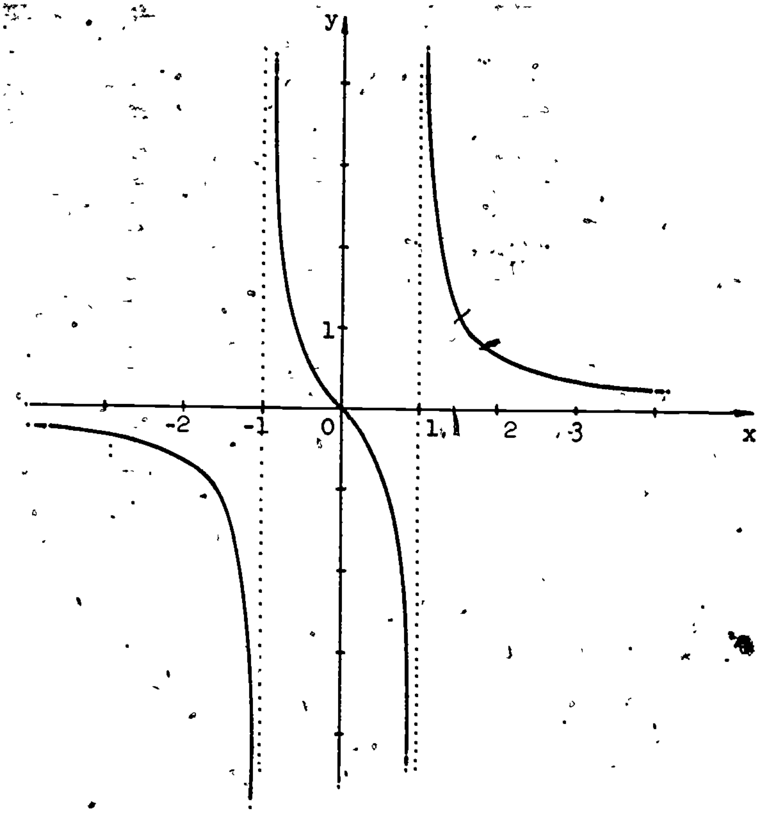
$$x^2 + 1 > 0 \text{ and } (x^2 - 1)^2 > 0.$$

Thus $f'(x) < 0$ for all $x \neq \pm 1$.

$$f''(x) = -(x^2 + 1) \cdot \frac{(-2)}{(x^2 - 1)^3} \cdot 2x + \frac{-(2x)}{(x^2 - 1)^2}$$

$$= \frac{4x(x^2 + 1) - 2x(x^2 - 1)}{(x^2 - 1)^3}$$

$$= \frac{2x(x^2 + 3)}{(x^2 - 1)^3}$$




$x^2 + 3 > 0$ for all values of $x$.

$(x^2 - 1)^3 < 0$ when $-1 < x \leq 1$

$(x^2 - 1)^3 \geq 0$ when $1 \leq |x|$

| | $x < -1$ | $-1 < x < 0$ | $0 < x < 1$ | $1 < x$ |
|---|---|---|---|---|
| $2x$ | - | - | + | + |
| $x^2 + 3$ | + | + | + | + |
| $(x^2 - 1)^2$ | + | - | - | + |
| $f''(x)$ | - | + | - | + |

$f$ is concave if $x < -1$

$f$ is convex if $-1 < x < 0$

$f$ is concave if $0 \leq x < 1$

$f$ is convex if $1 < x$.

(b) $f : x \to e^{1/x}$

$$f'(x) = e^{1/x} \cdot \left(-\frac{1}{x^2}\right)$$

$$= -\frac{e^{1/x}}{x^2}$$

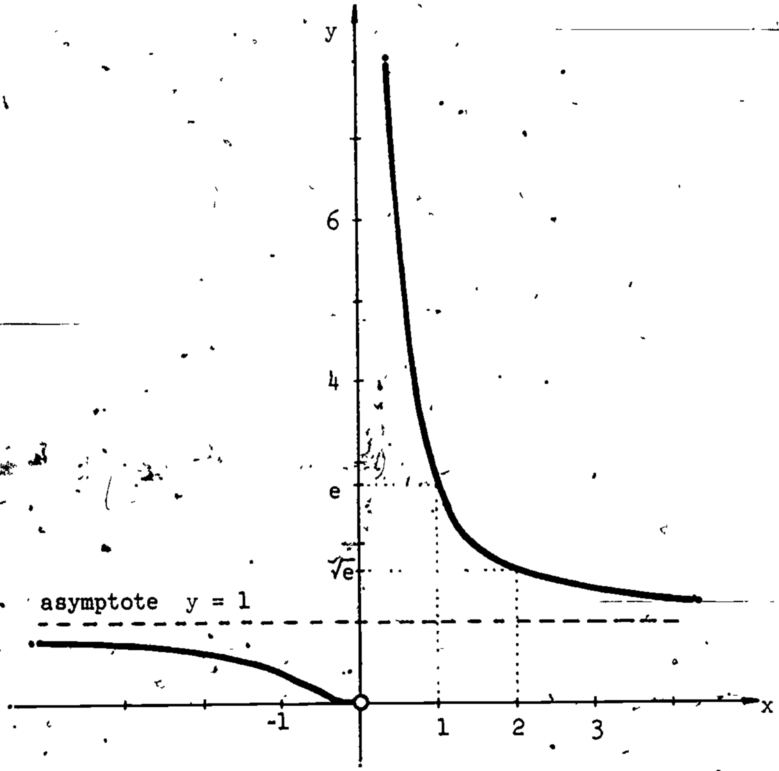
Thus $f'(x) < 0$ and $f$ is a decreasing function whenever $x \neq 0$.

$$f''(x) = -e^{1/x} \cdot \frac{-2}{x^3} + \frac{-1}{x^2} \cdot \left(-\frac{e^{1/x}}{x^2}\right)$$

$$= \frac{e^{1/x}}{x^4}(2x + 1)$$

$$f''(x) \geq 0 \quad \text{if} \quad 2x + 1 \geq 0 .$$

Thus $f$ is convex when $-\frac{1}{2} \leq x$ and $f$ is concave when $x < -\frac{1}{2}$.



(c) $f : x \to \log_e \dfrac{1 + x^2}{1 - x^2}$   $-1 < x < 1$

$$f'(x) = \frac{1 - x^2}{1 + x^2}\left[\frac{(1 - x^2)(2x) - (1 + x^2)(-2x)}{(1 - x^2)^2}\right]$$

$$= \frac{2x - 2x^3 + 2x + 2x^3}{(1 + x^2)(1 - x^2)}$$

$$= \frac{4x}{1 - x^4}$$

$f$ is increasing when $0 \le x < 1$.

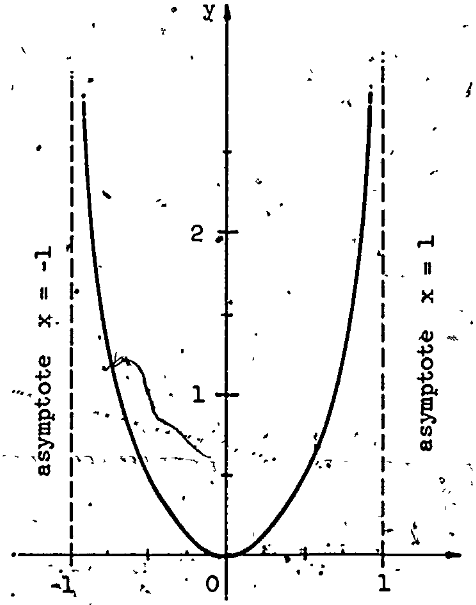
$f$ is decreasing when $-1 < x < 0$.

$$f''(x) = \frac{(1 - x^4)(4) - (4x)(-4x^3)}{(1 - x^4)^2}$$

$$= \frac{4 - 4x^4 + 16x^4}{(1 - x^4)^2}$$

$$= \frac{4 + 12x^4}{(1 - x^4)^2} > 0$$

$f$ is convex when $-1 < x < 1$.



8. (a) $y = xe^{-x^2}$, $x = 0$

$y' = x(e^{-x^2})(-2x) + e^{-x^2}$

$= e^{-x^2}(1 - 2x^2)$

$y'(0) = 1(1 - 0) = 1$

$y(0) = 0$

The tangent line at $(0,0)$ is $y = x$.

(b) $y = e^{-11x^2}$, $x = 1$

$y(1) = e^{-11}$

$y' = -22x\, e^{-11x^2}$

$y'(1) = -22e^{-11}$

The tangent line at $(1, e^{-11})$ is

$$y - e^{-11} = -22e^{-11}(x - 1)$$

$$y = -22e^{-11}x + 23e^{-11}$$

(c) $y = \sin(\pi - x^2)^{3/2}$, $x = \sqrt{\pi}$

$y(\pi) = 0$

$y' = \frac{3}{2}\sin(\pi - x^2)^{1/2} \cdot \cos(\pi - x^2) \cdot (-2x)$

$y'(\sqrt{\pi}) = 0$.

The tangent line at $(\sqrt{\pi}, 0)$ is $y = 0$.

(d) $y = \log_e(1 - x^2)$, $x = \frac{1}{2}$

$y(\frac{1}{2}) = \log_e(\frac{3}{4})$

$y' = \frac{1}{1 - x^2} \cdot (-2x)$

$y'(\frac{1}{2}) = -\frac{4}{3}$

The tangent line at $(\frac{1}{2}, \log_e \frac{3}{4})$ is

$$y - \log_e \frac{3}{4} = -\frac{4}{3}(x - \frac{1}{2}); \quad \log_e \frac{3}{4} \approx .2877$$

or $$y \approx -\frac{4}{3}x + .9544$$

(e) $y = e^{e^x}$, $x = 0$

$y(0) = e$

$y' = e^{e^x} \cdot e^x$

$y'(0) = e^1 \cdot 1 = e$

The tangent line at $(0,e)$ is

$$y - e = ex$$

or

$$y = ex + e.$$

(f). $y = (e^x)^\pi$, $x = e$

$y(e) = e^{\pi e}$

$y' = \pi e^{\pi x}$

$y'(e) = \pi e^{\pi e}$

The tangent line at $(e, e^{\pi e})$ is

$$y - e^{\pi e} = \pi e^{\pi e}(x - e)$$

9. $f(x) = (Ax + B)\sin x + (Cx + D)\cos x$ and $f'(x) = x \sin x$.

$$f'(x) = A \sin x + (Ax + B)\cos x + C \cos x - (Cx + D)\sin x$$

$$= (A - Cx - D)\sin x + (C + Ax + B)\cos x$$

Two conditions must be met.

(i) $(A - Cx - D)\sin x = x \sin x$

or $A - Cx - D = x$

(ii) $(C + Ax + B)\cos x = 0$

or $C + Ax + B = 0$

We first observe in (i) that $C = -1$ and that $A = D$. In (ii) $A = 0$ is obvious and also $C = -B$. Thus $A = 0$, $B = +1$, $C = -1$, $D = 0$. Then $f(x) = \sin x - x \cos x$.

10. $g(x) = (Ax^2 + Bx + C)\sin x + (Dx^2 + Ex + F)\cos x$ and $g'(x) = x^2 \cos x$

$$g'(x) = (Ax^2 + Bx + C)\cos x + (2Ax + B)\sin x - (Dx^2 + Ex + F)\sin x + (2Dx + E)\cos x$$

$$= (2Ax + B - Dx^2 - Ex - F)\sin x + (2Dx + E + Ax^2 + Bx + C)\cos x$$

$\therefore$ (i) $2Ax + B - Dx^2 - Ex - F = 0$

(ii) $2Dx + E + Ax^2 + Bx + C = x^2$

From (i) $D = 0$, $2A - E = 0$, and $B - F = 0$. Thus $D = 0$, $2A = E$, and $B = F$. Rewriting (ii) we have

$$2(0)x + 2A + Ax^2 + Bx + C = x^2.$$

It follows that $A = 1$, $B = 0$, and $C = -2A$. Thus $A = 1$, $B = 0$, $C = -2$, $D = 0$, $E = 2$, $F = 0$.

$$g(x) = (x^2 - 2)\sin x + 2x \cos x$$

11. $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$

$\dfrac{dy}{dx} = \cos x = \cos(t^2 + \frac{1}{t})$

$\dfrac{dx}{dt} = 2t - \dfrac{1}{t^2}$

$\therefore \dfrac{dy}{dt} = \cos(t^2 + \frac{1}{t})(2t - \frac{1}{t^2})$

$\left.\dfrac{dy}{dt}\right|_{t=1} = \cos 2$

$\left.\dfrac{dy}{dt}\right|_{t=2} = \dfrac{15}{4} \cos \dfrac{9}{2}$

12. $y = f(h(t))$

$\therefore \dfrac{dy}{dt} = f'(h(t))h'(t)$ and $\left.\dfrac{dy}{dt}\right|_{t=t_0} = f'(h(t_0))h'(t_0)$

13. $y = f(x)$ and $x = h(t)$ imply $\dfrac{dy}{dx} = f'(x)$, $\dfrac{dx}{dt} = h'(t)$,

$\dfrac{dy}{dt} = f'(h(t)) \cdot h'(t)$

$$\therefore \left.\frac{dy}{dx}\right|_{x=x_0} = f'(x_0) = f'(h(t_0)) = \frac{f'(h(t_0)) \cdot h'(t_0)}{h'(t_0)}$$

$$= \frac{\left.\frac{dy}{dt}\right|_{t=t_0}}{\left.\frac{dx}{dt}\right|_{t=t_0}}$$

14. (a) $D \sin x\big|_{x=0} + D \sin x\big|_{x=\pi/4} = \cos 0 + \cos \frac{\pi}{4} = 1 + \frac{\sqrt{2}}{2}$.

(b) $D(x^2 + \sin a \sin x)\big|_{x=5\pi/3} = 2(\frac{5\pi}{3}) + \sin a \sin \frac{5\pi}{3}$

$$= \frac{10\pi}{3} - \frac{\sqrt{3}}{2} \sin a.$$

(c) $\frac{d}{dx}(x^2 - a^2)\big|_{x=a} = 2a$

(d) $D(f(a) \sin x + f(x) \sin a + f(x) \sin x)\big|_{x=a}$

$$= f(a) \cos a + f'(a) \sin a + f'(a) \sin a + f(a) \cos a$$
$$= 2(f(a) \cos a + f'(a) \sin a).$$

15. $v = \frac{4}{3} \pi r^3$

$\therefore\ r = (\frac{3v}{4\pi})^{1/3}$

$v = 100t$

so $r = (\frac{300t}{4\pi})^{1/3}$.

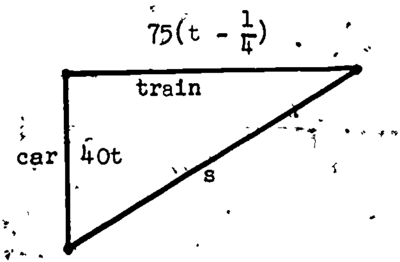
$$\therefore \frac{dr}{dt} = \frac{\frac{1}{3} \cdot \frac{300}{4\pi}}{(\frac{300t}{4\pi})^{2/3}}$$

When $r = 5$, $v = \frac{4}{3}\pi \cdot 5^3 = 100t$, so $t = \frac{5\pi}{3}$.

$$\therefore \frac{dr}{dt}\Big|_{t=5\pi/3} = \frac{\frac{100}{4\pi}}{\left(\frac{300 \cdot \frac{5\pi}{3}}{4\pi}\right)^{2/3}} = \frac{\frac{25}{\pi}}{(5^3)^{2/3}} = \frac{1}{\pi}$$

When the radius is 5 inches it is increasing at the rate of $\frac{1}{\pi}$ in./min.

16.



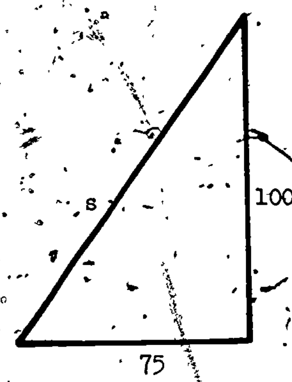

$$s = \sqrt{(40t)^2 + (72t - 18)^2}$$

$$\therefore \frac{ds}{dt} = \frac{\frac{1}{2}(2(40t)(40)) + 2(72t - 18)(72)}{[(40t)^2 + (72t - 18)^2]^{1/2}}$$

At $t = 1$,

$$\frac{ds}{dt}\Big|_{t=1} = \frac{40^2 + 54 \cdot 72}{(40^2 + 54^2)^{1/2}} = \frac{5488}{(4516)^{1/2}} \approx 81.9$$

Hence, the distance between the car and train is increasing at the rate of 81.9 mi./hr. one hour after the car crossed the intersection.

17. 


$$s = \sqrt{(75)^2 + (100t)^2}$$

$$\frac{ds}{dt} = \frac{\frac{1}{2} \cdot 2(100t)(100)}{[(75)^2 + (100t)^2]^{1/2}}$$

$$\left.\frac{ds}{dt}\right|_{t=3} = \frac{100^2 \cdot 3}{[75^2 + 300^2]^{1/2}} \approx \frac{30000}{309} \approx 97$$

The rocket is receding from the observer at the approximate rate of 97 ft./sec. 3 seconds after take-off.

Solutions Exercises 8-9

1. (a) $x \to \sqrt{\sin x}$

$$f'(x) = \frac{1}{2}(\sin x)^{-1/2} \cos x$$

or $$\frac{1}{2}\sqrt{\cot x \cos x}$$

(b) $x \to (\log_e x)^{\pi}$

$$f'(x) = \pi(\log_e x)^{\pi-1} \cdot \frac{1}{x}$$

$$= \frac{\pi}{x}(\log_e x)^{\pi-1}$$

(c) $s \to (s^3 + 3s)^{25}$

$$f'(s) = 25(s^3 + 3s)^{24} \cdot (3s^2 + 3)$$

(d) $t \to (e^t)^{-10}$

$$f'(t) = -10(e^t)^{-11} \cdot e^t$$

$$= -10e^{-10t}$$

(e) $x \to \dfrac{1}{\sqrt[3]{(1 - x)^2}} = (1 - x)^{-2/3}$

$$f'(x) = -\frac{2}{3}(1 - x)^{-5/3} \cdot (-1)$$

$$= \frac{2}{3}(1 - x)^{-5/3}$$

or $$= \frac{2}{3\sqrt[3]{(1 - x)^5}}$$

(f) $t \to (1 + \frac{1}{t})^{4/3}$

$$f'(t) = \frac{4}{3}(1 + \frac{1}{t})^{1/3} \cdot (-\frac{1}{t^2})$$

$$= -\frac{4}{3t^2}(1 + \frac{1}{t})^{1/3}$$

(g) $v \to \cos^{10} 2v$

$$f'(v) = 10\cos^9 2v \cdot (-\sin 2v) \cdot (2)$$
$$= -20\cos^9 2v \sin 2v$$

(h) $x \to \left(\int_0^x \sqrt{t^3+1}\,dt\right)^{1/2}$

$$F'(x) = \frac{1}{2}\left(\int_0^x \sqrt{t^3+1}\,dt\right)^{-1/2} \cdot \sqrt{x^3+1}$$

$$= \frac{\sqrt{x^3+1}}{2\left(\int_0^x \sqrt{t^3+1}\,dt\right)^{1/2}}$$

2. (a) $y = \dfrac{1}{1-x^2}$

$$y' = \frac{-(-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

(b) $y = \left(\dfrac{1}{1-x^2}\right)^5 = \dfrac{1}{(1-x^2)^5}$

$$y' = \frac{-5(1-x^2)^4(-2x)}{(1-x^2)^{10}}$$

$$= \frac{10x}{(1-x^2)^6}$$

(c) $y = \dfrac{1}{1+e^{2x}}$

$$y' = \frac{-2e^{2x}}{(1+e^{2x})^2}$$

(d) $y = \dfrac{1}{1+\log_e x}$

$$y' = \frac{-\frac{1}{x}}{(1+\log_e x)^2}$$

(e) $y = \dfrac{1}{\sqrt{x + \frac{1}{x}}}$

$$y' = \frac{-\frac{1}{2}\frac{1}{\sqrt{x + \frac{1}{x}}} \cdot \left(1 - \frac{1}{x^2}\right)}{x + \frac{1}{x}}$$

$$= \frac{1 - x^2}{2x^2\left(x + \frac{1}{x}\right)^{3/2}}$$

(f) $y = (\sin x + \cos x)^{-1}$

$$y' = -\frac{(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{\sin x - \cos x}{(\sin x + \cos x)^2}$$

3\. (a) $y = \sin^{3/2}(2x), \quad x = \frac{\pi}{6}$

$$y\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right)^{3/2}$$

$$y'(x) = \frac{3}{2}\sin^{1/2} 2x \cdot \cos 2x \cdot 2$$

$$= 3 \sin^{1/2} 2x \cos 2x$$

$$y'\left(\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2}\right)^{1/2} \cdot \frac{1}{2}$$

$$= \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right)^{1/2}$$

The tangent line at $\left(\frac{\pi}{6}, \left(\frac{\sqrt{3}}{2}\right)^{3/2}\right)$ is

$$y - \left(\frac{\sqrt{3}}{2}\right)^{3/2} = \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right)^{1/2}\left(x - \frac{\pi}{6}\right)$$

(b) $y = \left(\int_0^x e^{-t^2}\,dt\right)^2, \quad x = 0$

$y(0) = 0$

$y' = 2\left(\int_0^x e^{-t^2}\,dt\right)e^{-x^2}$

$y'(0) = 0$

The tangent line at $(0,0)$ is $y = 0$.

(c) $s = \sqrt{t + \frac{1}{t}}\,, \; t = 1$

$s(1) = \sqrt{2}$

$s' = \frac{1}{2}(t + \frac{1}{t})^{-1/2} \cdot (1 - \frac{1}{t^2})$

$s'(1) = 0$

The tangent line at $(1,\sqrt{2})$ is $s = \sqrt{2}$.

4. (a) $y = \dfrac{1}{1 + x^2}$

(i) $y$ is defined for all $x$.

(ii) $y' = \dfrac{-2x}{(1 + x^2)^2} = -2x(1 + x^2)^{-2}$

$y$ is increasing when $x \leq 0$.

$y$ is decreasing when $0 < x$.

(iii) $y'' = (-2x)(-2)(1 + x^2)^{-3}2x + -2(1 + x^2)^{-2}$

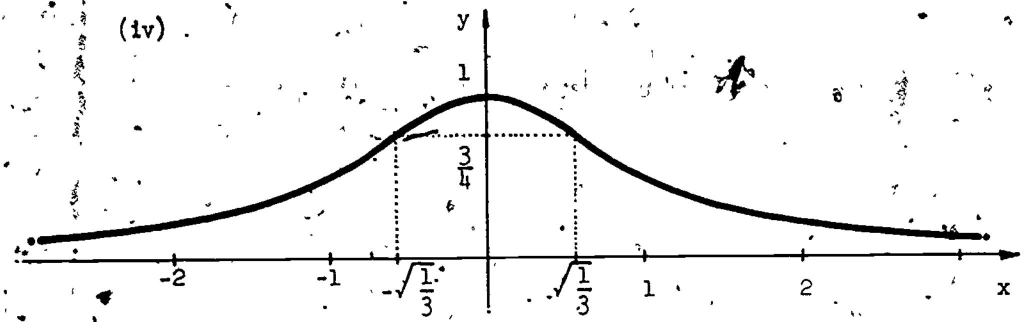
$= \dfrac{6x^2 - 2}{(1 + x^2)^3}$

$y$ is convex when $\sqrt{\frac{1}{3}} \leq |x|$

$y$ is concave when $|x| < \sqrt{\frac{1}{3}}$

(iv) Horizontal asymptote of $y = 0$.

(iv)




(b) $y = \sqrt{\sin x}$

(i) $y$ is defined when $2n\pi \le x \le (2n+1)\pi$, $n = 0, \pm 1, \pm 2, \ldots$

(ii) $y' = \frac{1 \cos x}{2\sqrt{\sin x}} = \frac{1}{2} \cos x \sin^{-1/2} x$

$y$ is increasing when $2n\pi \le x < (2n + \frac{1}{2})\pi$.

$y$ is decreasing when $(2n + \frac{1}{2})\pi \le x \le (2n + 1)\pi$.

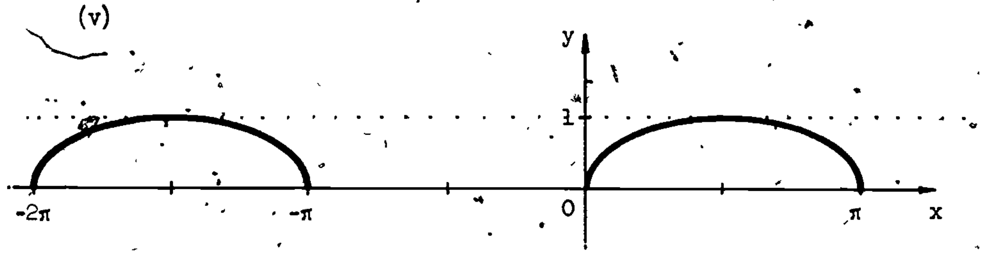
(iii) $y'' = \frac{1}{2}(\cos x)(-\frac{1}{2})(\sin^{-3/2} x \cos x) + \frac{1}{2}(\sin x)^{-1/2}(-\sin x)$

$= \frac{-(\cos^2 x + 2 \sin^2 x)}{4 \sin^{3/2} x}$

$y$ is everywhere concave.

(iv) There are no asymptotes.

(v)




5. (a) $x \to \frac{1}{1 - e^x}$, $x > 0$

$$f'(x) = \frac{-(-e^x)}{(1 - e^x)^2}$$

$$= \frac{e^x}{(1 - e^x)^2}$$

$f'(x) > 0$ thus $f$ is an increasing function.

(b) $x \to (x^3 + 3x)^{10}, \; x \geq 0$

$$f'(x) = 10(x^3 + 3x)^9(3x^2 + 3)$$
$$= 30(x^2 + 1)(x^3 + 3x)^9$$

$f'(x) \geq 0$ thus $f$ is an increasing function.

6. (a) $y = \sec x = \dfrac{1}{\cos x}$

$$y' = \frac{-(-\sin x)}{\cos^2 x}$$
$$= \tan x \sec x$$

(b) $y = \csc x = \dfrac{1}{\sin x}$

$$y' = \frac{-\cos x}{\sin^2 x}$$
$$= -\cot x \csc x$$

(c) $y = \tan x = \dfrac{\sin x}{\cos x} = \sin x(\cos x)^{-1}$

$$y' = \sin x(-1)(\cos x)^{-2}(-\sin x) + \cos x(\cos x)^{-1}$$
$$= \frac{\sin^2 x}{\cos^2 x} + 1$$
$$= \tan^2 x + 1 = \sec^2 x$$

(d) $y = \cot x = \dfrac{\cos x}{\sin x} = \cos x(\sin x)^{-1}$

$$y' = \cos x(-1)(\sin x)^{-2}\cos x + (-\sin x)(\sin x)^{-1}$$
$$= -\frac{\cos^2 x}{\sin^2 x} - 1$$
$$= -(\cot^2 x + 1)$$
$$= -\csc^2 x$$

(e) $D(\tan 3x) = 3 \sec^2 3x$

(f) $D\sqrt{\tan 2x} = \dfrac{2 \sec^2 2x}{2\sqrt{\tan 2x}}$

(g) $D(\sec^2 x^2) = 2 \sec x^2 \cdot (\sec x^2 \tan x^2)(2x)$

$$= 4x \sec^2 x^2 \tan x^2$$

(h) $D(\csc 3x)^{1/6} = \frac{1}{6}(\csc 3x)^{-5/6} \cdot (-\csc 3x \cot 3x)(3)$

$$= -\frac{1}{2}(\csc 3x)^{1/6} \cot 3x$$

(i) $D[\sec(\csc x)] = [\sec(\csc x)\tan(\csc x)][-\csc x \cot x]$

7. $f : x \to \sec x$; $f$ is not defined for $x = (n + \frac{1}{2})\pi$, $n = 0, \pm 1, \pm 2, \ldots$

$f' : x \to \sec x \tan x = \dfrac{\sin x}{\cos^2 x}$

$f'(x) \geq 0$ when $\sin x > 0$ or when $2n\pi \leq x < (2n + 1)\pi$, $x \neq (n \pm \frac{1}{2})\pi$

$f''(x) = \sec x \cdot \sec^2 x + \tan x \sec x \tan x$

$$= \sec x (\sec^2 x + \tan^2 x) = \frac{\sec^2 x + \tan^2 x}{\cos x}$$

$f''(x) \geq 0$ when $\cos x \geq 0$.

Thus $f$ is convex when $(2n - \frac{1}{2})\pi < x < (2n + \frac{1}{2})\pi$.

8. (a) $D(\sec x \csc x) = -\sec x \cot x \csc x + \csc x \tan x \sec x$

$$= -\sec x \frac{\csc x}{\sec x} \csc x + \csc x \frac{\sec x}{\csc x} \sec x$$

(i) $= -\csc^2 x + \sec^2 x$

(ii) $= -1 - \cot^2 x + 1 + \tan^2 x$

$= -\cot^2 x + \tan^2 x$

(iii) $= -4 \csc 2x \cot 2x$

(b) (i) $D(\tan x \cot x) = D(1) = 0$

(ii) $D(\sin x \csc x) = D(1) = 0$

(iii) $D(\cos x \cdot \sec x) = D(1) = 0$

(c) (i) $D(\sin x \cot x) = D(\cos x)$

$= -\sin x$

(ii) $D(\cos x \tan x) = D(\sin x)$

$= \cos x$

9. (a) $D\left(\frac{\tan^{(k+1)} x}{k+1}\right) = (k+1) \cdot \frac{\tan^{((k+1)-1)} x}{k+1} \cdot D(\tan x)$

$$= \tan^k x \sec^2 x$$

(b) $D(\frac{1}{k} \csc^k x) = k[\frac{1}{k} \csc^{(k-1)} x]D(\csc x)$

$$= \csc^{(k-1)} x \csc x \cot x$$

$$= \csc^k x \cot x$$

(c) $D(\cot^2 x) = D(\csc^2 x - 1)$

$$= D(\csc^2 x) + D(-1)$$

$$= D(\csc^2 x)$$

10. (a) $\frac{u}{v} = u \cdot \frac{1}{v}$

$$(\frac{u}{v})' = u \cdot (\frac{1}{v})' + u'(\frac{1}{v})$$

$$= u \cdot \frac{-v'}{v^2} + u' \cdot \frac{1}{v}$$

$$= \frac{vu' - uv'}{v^2}$$

(b) $D\left(\frac{x^2+1}{3x^2-x}\right) = \frac{(3x^2-x)(2x) - (x^2+1)(6x-1)}{(3x^2-x)^2}$

$$= \frac{-x^2 - 6x + 1}{(3x^2-x)^2}$$

1. (a) $D\left(\frac{x}{x-1}\right) = \frac{(x-1)(1) - (x)(1)}{(x-1)^2}$

$= \frac{-1}{(x-1)^2}$

(b) $D\left(\frac{x^2}{1+x^2}\right) = \frac{(1+x^2)(2x) - (x^2)(2x)}{(1+x^2)^2}$

$= \frac{2x}{(1+x^2)^2}$

(c) $D\left(1 - \frac{1}{x}\right)^{-1} = D\left(\frac{x}{x-1}\right)$

$= \frac{-1}{(x-1)^2}$ from part (a).

(d) $D\left(\frac{3+2x^2}{2-x^2}\right) = \frac{(2-x^2)(4x) - (3+2x^2)(-2x)}{(2-x^2)^2}$

$= \frac{14x}{(2-x^2)^2}$

(e) $D\left(\frac{1}{x} + \frac{1}{1-x}\right) = D\left(\frac{1-x+x}{x-x^2}\right)$

$= D\left(\frac{1}{x-x^2}\right)$

$= \frac{-(1-2x)}{(x-x^2)^2}$

$= \frac{2x-1}{(x-x^2)^2}$

(f) $D\left(\frac{\sqrt{x}}{1+x^2}\right) = \frac{(1+x^2)\frac{1}{2\sqrt{x}} - \sqrt{x}(2x)}{(1+x^2)^2}$

$= \frac{1+x^2-4x^2}{2\sqrt{x}\,(1+x^2)^2}$

$= \frac{1-3x^2}{2\sqrt{x}\,(1+x^2)^2}$

(g) $$D\left(\frac{1}{1+\sqrt{x}}\right) = \frac{-\frac{1}{2\sqrt{x}}}{(1+\sqrt{x})^2}$$

$$= \frac{-1}{2\sqrt{x}\,(1+\sqrt{x})^2}$$

(h) $$D\left(\frac{x^2-1}{x^2+1}\right)^{-1} = D\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}$$

(i) $$D\left(\frac{\sin x}{1+\tan x}\right) = \frac{(1+\tan x)\cos x - \sin x \sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\cos x + \sin x - \sin x \sec^2 x}{(1+\tan x)^2}$$

(j) $$D\left(\frac{e^x}{1+x^2}\right) = \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2}$$

$$= \frac{e^x(x^2-2x+1)}{(1+x^2)^2}$$

$$= \frac{e^x(x-1)^2}{(1+x^2)^2}$$

(k) $$D\left(\frac{x\log_e x}{1-2x}\right) = \frac{(1-2x)(x\cdot\frac{1}{x}+\log_e x)-(x\log_e x)(-2)}{(1-2x)^2}$$

$$= \frac{1-2x+\log_e x}{(1-2x)^2}$$

($\ell$) $D(\cos x \sec x) = D(1) = 0$

(m) $$D\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right) = \frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$

$$= \frac{4}{(e^x+e^{-x})^2}$$

(n) $D[(1 + \frac{1}{x})(1 + \log_e x)] = (1 + \frac{1}{x})\frac{1}{x} + (1 + \log_e x)\left(-\frac{1}{x^2}\right)$

$$= (1 + \frac{1}{x})\frac{1}{x} + (1 + \log_e x)\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^2} - \frac{1}{x^2}\log_e x$$

$$= \frac{1}{x^2}(x - \log_e x)$$

(o) $D\left(\frac{\log_e x^2}{\sqrt{x^2 + 1}}\right) = D\left(\frac{2 \log_e x}{\sqrt{x^2 + 1}}\right)$

$$= \frac{\sqrt{x^2 + 1} \cdot \frac{2}{x} - (2 \log_e x) \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x}{x^2 + 1}$$

$$= \frac{\frac{2}{x\sqrt{x^2 + 1}}[(x^2 + 1) - x^2 \log_e x]}{(x^2 + 1)}$$

$$= \frac{2(x^2 + 1 - x^2 \log_e x)}{x(x^2 + 1)^{3/2}}$$

2. $D(\cot x) = D(\frac{\cos x}{\sin x})$

$$= \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\csc^2 x$$

3. (a) $y = \frac{x + 2}{x^2 - 1}$

This function is not defined for $x = \pm 1$. To the left of $x = -1$, $y \to +\infty$ and to the right of $x = -1$, $y \to -\infty$. To the left of $x = +1$, $y \to -\infty$ and to the right of $x = +1$, $y \to +\infty$. This describes the vertical asymptotes at $x = +1$ and $x = -1$. At $x = -2$, $y = 0$ and at $x = 0$, $y = -2$.

$$y = \frac{\frac{x}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$

$$= \frac{\frac{1}{x} + \frac{2}{x^2}}{1 - \frac{1}{x^2}}$$

As $x \to +\infty$, $y \to 0^+$ and as $x \to -\infty$, $y \to 0^-$. This describes the horizontal asymptote, $y = 0$.

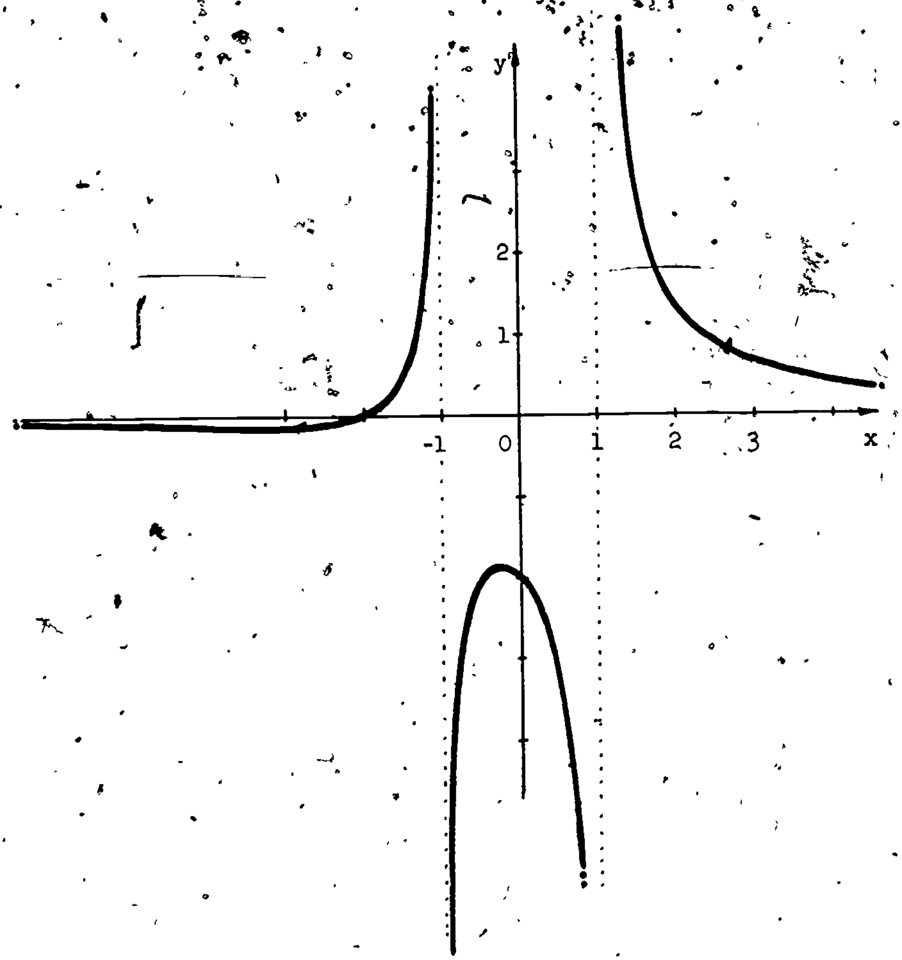
$$y' = \frac{(x^2 - 1)(1) - (x + 2)(2x)}{(x^2 - 1)^2}$$

$$= \frac{-x^2 - 4x - 1}{(x^2 - 1)^2} = \frac{-(x^2 + 4x + 1)}{(x^2 - 1)^2}$$

$$y' = 0 \text{ if } x = \frac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{3}.$$

There are two extremum points, one is a minimum point when $x = -2 - \sqrt{3} \approx -3.732$ and the other is a maximum point when $x = -2 + \sqrt{3} \approx -.268$. We see that $y$ decreases for $x < -2 - \sqrt{3}$, increases for $-2 - \sqrt{3} \leq x < -2 + \sqrt{3}$, and finally decreases again for $-2 + \sqrt{3} \leq x$.

The discussion of $y''$ is rather tedious and does not greatly increase our understanding of the function.




(b) $y = \frac{x - 1}{x + 1}$

$y$ is undefined for $x = -1$. As $x \to -1$ from the left $y \to +\infty$ and as $x \to -1$ from the right $y \to -\infty$. This describes the vertical asymptote at $x = -1$.

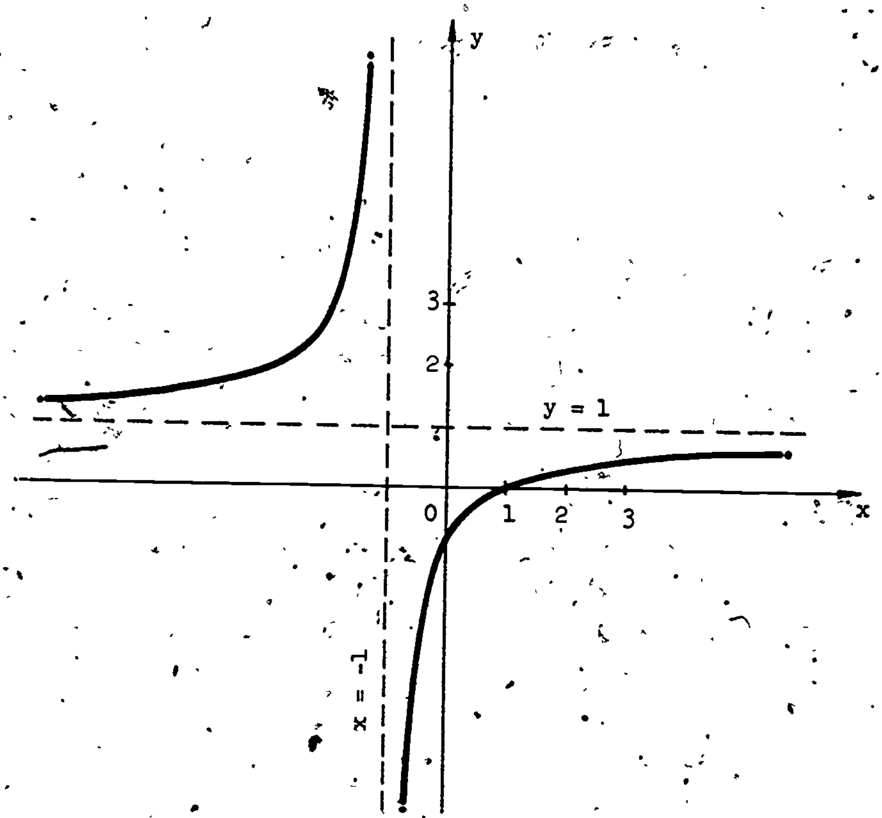
There is a zero at $x = 1$.

Rewriting $y = \dfrac{1 - \frac{1}{x}}{1 + \frac{1}{x}}$ we see that as $x \to +\infty$, $y \to 1$ and as $x \to -\infty$, $y \to 1$. Thus, there is a horizontal asymptote of $y = 1$.

$$y' = \frac{(x + 1)(1) - (x - 1)(1)}{(x + 1)^2}$$

$$= \frac{2}{(x + 1)^2}$$

$y'$ is always positive and $y$ is increasing.




(c) $y = \frac{e^{-2x}}{1 + x}$, $y$ is undefined at $x = -1$.

As $x \to -1$ from the left $y \to -\infty$ and as $x \to -1$ from the right $y \to +\infty$. This describes the vertical asymptote at $x = -1$.

When $x \to +\infty$, $y \to \frac{0}{\infty} = 0$.

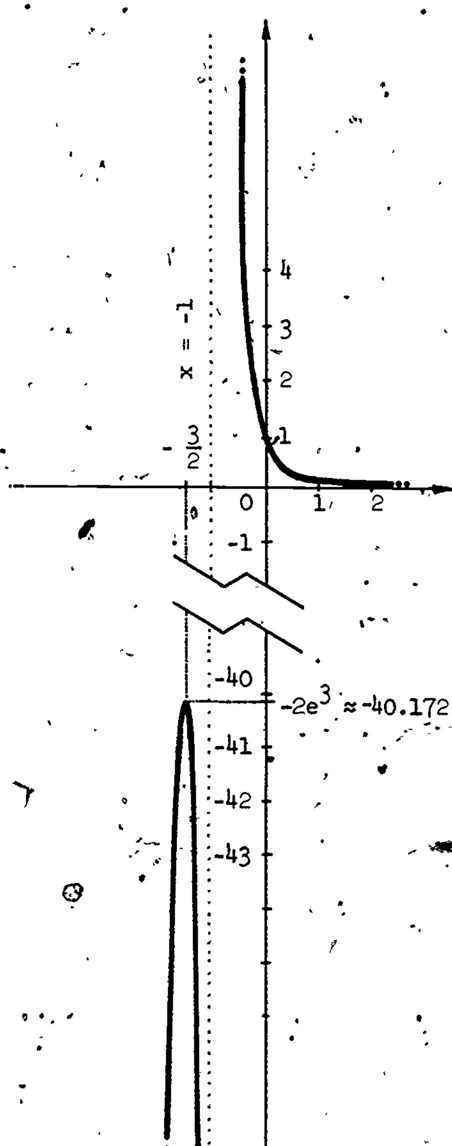
From previous discussions in Chapter 5 we know that $\frac{e^{|x|}}{|x|}$ is greater than any predetermined number when $x$ is large enough. It follows that as $x \to -\infty$ we have $e^{-2x} > 0$ and $1 + x < 0$, thus, $y \to -\infty$. There is an asymptote for $x \to +\infty$ which is $y = 0$, but none for $x \to -\infty$.

$$y' = \frac{(1 + x)(-2e^{-2x}) - e^{-2x}}{(1 + x)^2}$$

$$= \frac{-e^{-2x}(3 + 2x)}{(1 + x)^2}$$

$$y' \geq 0 \text{ if } 3 + 2x \leq 0 \text{ or } x \leq -\frac{3}{2}$$

Thus $y$ is increasing when $-x \leq -\frac{3}{2}$ and decreasing when $-\frac{3}{2} < x$. It appears that $x = -\frac{3}{2}$ is a maximum.




4. (a) $\int_0^{\pi/4} \sec^2 x \, dx = \tan x \Big|_0^{\pi/4}$

$= 1 - 0 = 1$

(b) $\int_{-\pi/3}^{0} \sec x \tan x \, dx = \sec x \Big|_{-\pi/3}^{0}$

$= 1 - (+2) = -1$

5. (a) $z = \frac{y}{w}$

$$\therefore \frac{dz}{dt} = \frac{w \cdot \frac{dy}{dt} - y \cdot \frac{dw}{dt}}{[\frac{dw}{dt}]^2},$$

and $$\frac{dz}{dx} = \frac{dt}{dx} \cdot \frac{dz}{dt} = \frac{dt}{dx} \cdot \frac{w \cdot \frac{dy}{dt} - y \cdot \frac{dw}{dt}}{[\frac{dw}{dt}]^2}.$$

(b) $y = f(t)$, $w = g(t)$, and $t = h(x)$ implies

$$\frac{dy}{dt} = f'(t), \quad \frac{dw}{dt} = g'(t), \quad \text{and} \quad \frac{dt}{dx} = h'(x),$$

so

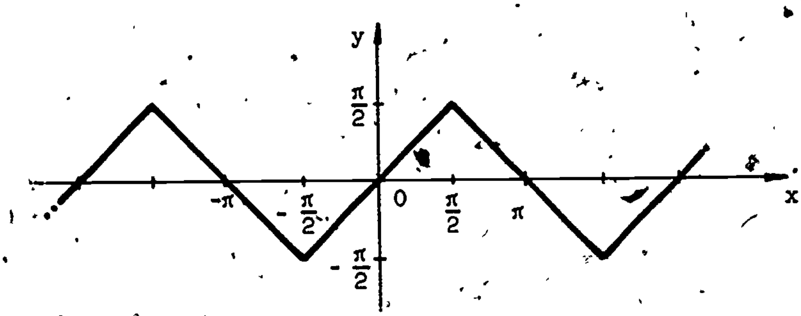
$$\begin{aligned}\frac{dz}{dx} &= h'(x) \cdot \frac{g(t) \cdot f'(t) - f(t)g'(t)}{[g'(t)]^2} \\ &= h'(x) \cdot \frac{g(h(x))f'(h(x))h'(x) - f(h(x))g'(h(x))h'(x)}{[g'(h(x))h'(x)]^2} \\ &= \frac{g(h(x))f'(h(x)) - f(h(x))g'(h(x))}{[g'(h(x))]^2}.\end{aligned}$$

## Solutions Exercises 8-11

1. (a) $f : x \rightarrow \arcsin(\sin x)$

Domain: the set of all real numbers.

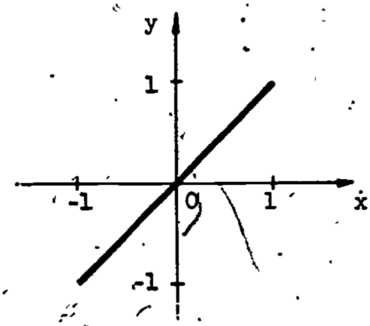
Range: the set of all real numbers $y = f(x)$ such that $-\frac{\pi}{2} \leq y \leq 2$.




(b) $f(x) = \sin(\arcsin x)$

Domain: all $x$ such that $-1 \leq x \leq 1$.

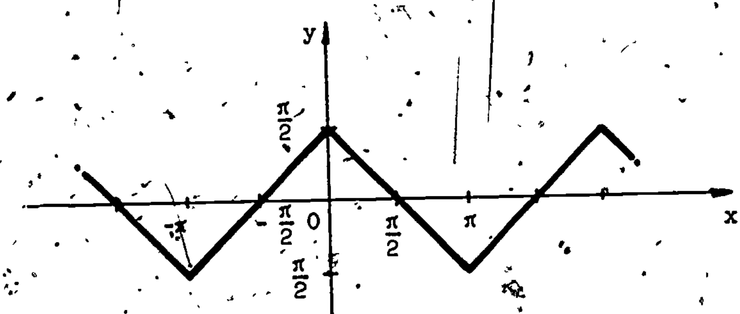
Range: all $y$ such that $-1 \leq y \leq 1$.




(c) $f(x) = \arcsin(\cos x)$

Domain: set of all real numbers.

Range: all $y = f(x)$ where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

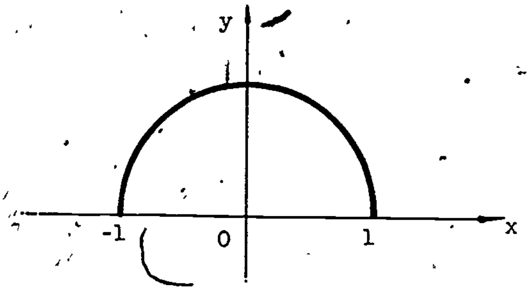


(d) $f(x) = \cos(\arcsin x)$

Domain: all $x$ where $-1 \le x \le 1$.

Range: all $y = f(x)$ where $0 \le y \le 1$.

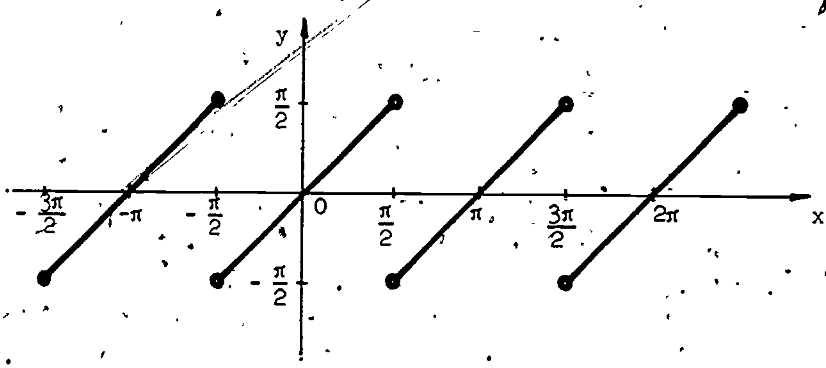
The graph is the semicircle with center at origin, radius = 1, and $y \ge 0$.




(e) $f(x) = \arctan(\tan x)$

Domain: all real $x$ except $x = (2n + 1)\frac{\pi}{2}$, $n$ an integer.

Range: all $y = f(x)$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$.




2. Let $g : x \to \arccos x$ and $f : y \to \cos y$, where $y = \arccos x$, $-1 \le x \le 1$, $0 \le y \le \pi$.

Then $g'(x) = \frac{1}{f'(y)}$ (for $f'(y) \ne 0$, i.e., for $-\sin y \ne 0$) for $0 < y < \pi$ and hence for $-1 < x < 1$.

$\therefore$ $D \arccos x = g'(x) = \frac{1}{-\sin y}$, where $\sin y > 0$,

$$= \frac{-1}{\sqrt{1 - x^2}}, \ -1 < x < 1.$$

3. (a) Let $g : x \to \operatorname{arccot} x$ and $f : y \to \cot y$ where $y = \operatorname{arccot} x$ for all real $x$ and $0 < y < \pi$.

$$g'(x) = \frac{1}{f'(y)} = \frac{1}{-\csc^2 y} = \frac{-1}{1 + \cot^2 y}, \quad 0 < y < \pi,$$

$$= \frac{-1}{1 + x^2}, \quad \text{for all } x.$$

(b) Let $g : x \to \operatorname{arcsec} x$ and $f : y \to \sec y$ where $y = \operatorname{arcsec} x$, $|x| \geq 1$ and $0 \leq y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y \leq \pi$.

Then $g'(x) = \dfrac{1}{f'(y)}$

$$= \frac{1}{\sec y \tan y}, \quad 0 < y < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < y < \pi$$

$$= \frac{1}{|x|\sqrt{x^2 - 1}}, \quad |x| > 1$$

Note: $\dfrac{1}{\sec y \tan y} = \dfrac{\cos^2 y}{\sin y} > 0$ for $0 < y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y < \pi$, hence $g'(x) > 0$ for all $x$ such that $|x| > 1$.

(c) Let $g(x) = \operatorname{arccsc} x$ and $f(y) = \csc y$ where $y = \operatorname{arccsc} x$ for $|x| \geq 1$ and $0 < |y| \leq \frac{\pi}{2}$.

Then $g'(x) = \dfrac{1}{f'(y)}$

$$= \frac{1}{-\csc y \cot y}, \quad 0 < |y| < \frac{\pi}{2}$$

$$= \frac{-1}{|x|\sqrt{x^2 - 1}}, \quad |x| > 1.$$

Note: $g'(x) < 0$ since $\dfrac{1}{\csc y \cot y} = \dfrac{-\sin^2 y}{\cos y} < 0$ for $0 < |y| < \frac{\pi}{2}$.

4. (a) $D(\arcsin x + \arccos x) = \dfrac{1}{\sqrt{1 - x^2}} + \dfrac{-1}{\sqrt{1 - x^2}} = 0$

Note that $\arcsin x + \arccos x = \frac{\pi}{2}$ and $D(\frac{\pi}{2}) = 0$.

(b) $D(x^2 \arcsin x) = x^2 \cdot \dfrac{1}{\sqrt{1 - x^2}} + 2x \arcsin x$

$$= x\left(\frac{x}{\sqrt{1 - x^2}} + 2 \arcsin x\right)$$

(c) $D\left(\dfrac{x^2}{\arctan x}\right) = \dfrac{2x \arctan x - \dfrac{x^2}{1 + x^2}}{(\arctan x)^2}$

$$= \frac{2x(1 + x^2) \arctan x - x^2}{(1 + x^2)(\arctan x)^2}$$

(d) $D(\arcsin x)^3 = \dfrac{3(\arcsin x)^2}{\sqrt{1 - x^2}}$

(e) $D\left(\dfrac{1}{1 + \arcsin x}\right) = \dfrac{-1}{\sqrt{1 - x^2}(1 + \arcsin x)^2}$

(f) $D\left(\dfrac{1 - \arctan x}{1 + \arctan x}\right) = \dfrac{(1 + \arctan x)\dfrac{-1}{1 + x^2} - (1 - \arctan x)\dfrac{1}{1 + x^2}}{(1 + \arctan x)^2}$

$$= \frac{-2}{(1 + x^2)(1 + \arctan x)^2}$$

5. Let $f : x \to \arcsin x$.

Then $\dfrac{f(x + h) - f(x)}{h}$, (for $x = 0$)

$$= \frac{\arcsin h - \arcsin 0}{h}$$

$$= \frac{\arcsin h}{h}.$$

$\therefore \lim\limits_{h \to 0} \dfrac{\arcsin h}{h} = f'(0) = \dfrac{1}{\sqrt{1 - 0^2}} = 1.$

6. (a) $y = \arcsin x^2$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{1 - x^4}}$$

(b) $y = \arctan (3x + 2)$

$$\frac{dy}{dx} = \frac{3}{9x^2 + 12x + 5}$$

(c) $y = e^{\arcsin x}$

$$\frac{dy}{dx} = \frac{e^{\arcsin x}}{\sqrt{1 - x^2}}$$

(d) $y = e^{2x} \arcsin \frac{1}{x}$

$$\frac{dy}{dx} = e^{2x} \cdot \frac{-1}{\sqrt{1 - \frac{1}{x^2}}} \cdot \frac{-1}{x^2} + 2e^{2x} \arcsin \frac{1}{x}$$

$$= e^{2x} \left(\frac{-1}{x^2 \sqrt{x^2 - 1}} + 2 \arcsin \frac{1}{x}\right)$$

7. (a) $\displaystyle\int_0^1 \frac{1}{1 + x^2}\, dx = \arctan x \Big|_0^1$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

(b) $\displaystyle\int_{-\pi/4}^{\pi/6} \frac{1}{\sqrt{1 - t^2}}\, dt = \arcsin t \Big|_{-\pi/4 \approx -0.7854}^{\pi/6 \approx 0.5236}$

$$\approx \arcsin(0.5236) - \arcsin(-0.7854)$$

$$\approx 0.551 - (-0.895) \approx 1.45$$

8. (a) $\displaystyle F(x) = \int_0^x \frac{2}{1 + t^2}\, dt$

$$F'(x) = \frac{2}{1 + x^2}$$

(b) $F(x) = \int_0^{x^3} \frac{3}{\sqrt{1 - t^2}}\, dt$

$F'(x) = \frac{9x^2}{\sqrt{1 - x^6}}$

(c) $F(x) = \int_0^{\sin x} \frac{1}{1 + t^2}\, dt$

$F'(x) = \frac{\cos x}{1 + \sin^2 x}$

9. $\lim_{n \to \infty} \int_0^n \frac{1}{1 + t^2}\, dt = \lim_{n \to \infty} \arctan \Big|_0^n$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$

10. (a) $g : x \to \frac{1 - x}{1 + x}, \quad x > -1$

$f : x \to \frac{1 - x}{1 + x}$

$f'(x) = \frac{(1 + x)(-1) - (1 - x)(1)}{(1 + x)^2}$

$f'(x) = \frac{-2}{(1 + x)^2}$

(b) $g : x \to x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

$f : x \to \frac{|x|}{x} \sqrt{|x|} = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$

$f' : x = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } x \geq 0 \\ -\frac{-1}{2\sqrt{-x}} & \text{if } x < 0 \end{cases}$

$f' : x = \frac{1}{2\sqrt{|x|}}$

11. If $f$ and $g$ are inverses let $g(x) = c$. But then $f(c) = x$ or by substitution $f(g(x)) = x$.

The derivative $D(f(g(x))) = f'(g(x))g'(x) = 1$.

Then $$g'(x) = \frac{1}{f'(g(x))}.$$

The only difference is that rule (5) was developed for a strictly increasing function, that is $f'(g(x)) > 0$.

12. $f_1$ and $f_2$ are the inverses of $g_1$ and $g_2$ respectively and $g(x) = g_1(g_2(x))$.

(a) Since $f_1(g_1(x)) = x$

then
$$f_1(g(x)) = f_1(g_1(g_2(x))) = g_2(x).$$

Since $f_2(g_2(x)) = x$

then
$$f_2(f_1(g(x))) = f_2(g_2(x)) = x.$$

It follows that $f_2(f_1(x))$ is the inverse of $g(x)$.

(b) $x \rightarrow (3x + 2)^2, \; x \geq -\frac{2}{3}$.

Let $g_1 : x \rightarrow x^2$ and $g_2 : x \rightarrow 3x + 2$.

We find that $f_1 : x \rightarrow \sqrt{x}$ and $f_2 : x \rightarrow \frac{x - 2}{3}$

Finally,
$$f(x) = f_2(f_1(x)) = f_2(\sqrt{x}) = \frac{\sqrt{x} - 2}{3}.$$

(c)
$$f'(x) = D\left(\frac{\sqrt{x}}{3} - \frac{2}{3}\right) = \frac{1}{6\sqrt{x}}$$

13. f and g are inverses.

$$y = g(x) \text{ and } x = f(y)$$

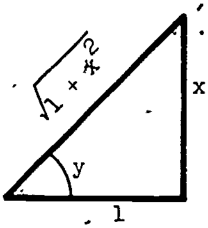
$$\left.\frac{dx}{dy}\right|_{y=a} = f'(a)$$

$$\left.\frac{dy}{dx}\right|_{x=f(a)} = g'(f(a)) = \frac{1}{f'(a)}, \qquad \text{by (5).}$$

Thus
$$\left.\frac{dx}{dy}\right|_{y=a} = \frac{1}{\left.\frac{dy}{dx}\right|_{x=f(a)}}.$$

14. (a) $y = \arctan x$ means that $x = \tan y$.

Thus $\frac{dx}{dy} = \sec^2 y$ and $\frac{dy}{dx} = \frac{1}{\sec^2(\arctan x)}$




$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

(b) If $y = \log_e x$ then $x = e^y$. Thus $\frac{dx}{dy} = e^y = e^{\log_e x} = x$ and $\frac{dy}{dx} = \frac{1}{x}$.

(c) If $y = \sqrt{x}$, then $x = y^2$. Then $\frac{dx}{dy} = 2y = 2\sqrt{x}$ and $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

(d) If $y = x^\pi$ then $x = \sqrt[\pi]{y}$ provided that $x \geq 0$.

Thus $\frac{dx}{dy} = \frac{1}{\pi} y^{(1/\pi - 1)} = \frac{1}{\pi}(x^\pi)^{(1/\pi - 1)} = \frac{1}{\pi} x^{1-\pi}$

and $\frac{dy}{dx} = \frac{\pi}{x^{1-\pi}} = \pi x^{\pi - 1}$.

## Solutions Exercises 8-12

1. $y = x^r$, where $r = \frac{p}{q}$, $x > 0$. If $y^q = x^p$, $Dy^q = Dx^p$ and

$$qy^{q-1}\,Dy = px^{p-1}, \text{ whence } Dy = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\,x^{p/q\,-1} = rx^{r-1}.$$

2. (a) $5x^2 + y^2 = 12,$

$10x + 2y\,y' = 0$ and $y' = \frac{-5x}{y}$

(b) $2x^2 - y^2 + x - 4 = 0$

$4x - 2y\,y' + 1 = 0$

$y' = \frac{4x + 1}{2y}$

(c) $y^2 - 3x^2 + 6y = 12$

$2y\,y' - 6x + 6\,y' = 0$

$y' = \frac{3x}{y + 3}$

(d) $x^3 + y^3 - 2xy = 0$

$3x^2 + 3y^2\,y' - 2x\,y' - 2y = 0$

$y' = \frac{2y - 3x^2}{3y^2 - 2x}$

3. (a) $x^2 = \frac{y - x}{y + x}$

$$2x = \frac{(y + x)(Dy - 1) - (y - x)(Dy + 1)}{(y + x)^2}$$

$$Dy = \frac{x(y + x)^2 + y}{x}$$

(b) $x^2y + xy^2 = x^3$

$2xy + x^2\,Dy + y^2 + 2xy\,Dy = 3x^2$

$$Dy = \frac{3x^2 - 2xy - y^2}{x^2 + 2xy}$$

(c) $x^m y^n = 10$, (m, n integers)

$$mx^{m-1} y^n + nx^m y^{n-1} Dy = 0$$

$$Dy = \frac{-my}{nx}$$

(d) $\sqrt{xy} + x = y^{-1}$

$$\frac{1}{2}(xy)^{-1/2}(x\,Dy + y) + 1 = -\frac{Dy}{y^2}$$

$$Dy = \frac{-2y^2\sqrt{xy} - y^3}{xy^2 + 2\sqrt{xy}}$$

4. (a) $2x^2 + 3xy + y^2 + x - 2y + 1 = 0$ at the point $(-2,1)$.

$$4x + 3y + 3x\,y' + 2y\,y' + 1 - 2y' = 0$$

At $(-2,1)$, $y' = -\frac{2}{3}$.

(b) $x^3 + y^2x^2 + y^3 - 1 = 0$ at the point $(1,-1)$.

$$3x^2 + 2yx^2\,y' + 2xy^2 + 3y^2\,y' = 0$$

At $(1,-1)$, $y' = -5$.

(c) $x^2 - x\sqrt{xy} - 6y^2 = 2$ at the point $(4,1)$.

$$2x - \frac{3}{2}\sqrt{xy} - \frac{1}{2}x^{3/2}y^{-1/2}\,y' - 12y\,y' = 0$$

At $(4,1)$, $y' = \frac{5}{16}$.

(d) $x \cos y = 3x^2 - 5$ at the point $(\sqrt{2}, \frac{\pi}{4})$

$$x(-\sin y\,y') + \cos y = 6x$$

At $(\sqrt{2}, \frac{\pi}{4})$, $y' = -\frac{11\sqrt{2}}{2}$

5. (a) $x^3 - 3axy + y^3 = 0$

$3x^2 - 3ax\,Dy - 3ay + 3y^2\,Dy = 0$

$$Dy\Big|_{x=y} = \left.\frac{-3x^2 + 3ay}{-3ax + 3y^2}\right|_{x=y\neq 0} = -1$$

(b). $x^m + y^m = 2$

$mx^{m-1} + my^{m-1}\,Dy = 0$

$$Dy\Big|_{x=y} = \left.\frac{-mx^{m-1}}{my^{m-1}}\right|_{x=y\neq 0} = -1$$

(c) $x^2 + y^2 = 2axy + a^2$

$2x + 2y\,Dy = 2ax\,Dy + 2ay$

$$Dy\Big|_{x=y} = \left.\frac{2ay - 2x}{2y - 2ax}\right|_{x=y} = -1$$

All three curves are symmetric about the line $y = x$. Thus at the point where $x = y$ the tangents to the curves are orthogonal to the line.

6. (a) $a \sin y + b \cos x = 0$ ($a$, $b$ constant).

$a \cos y\, y' - b \sin x = 0$

$$y' = \frac{b \sin x}{a \cos y}.$$

(b) $x \cos y + y \sin x = 0$

$\cos y - x \sin y\, y' + \sin x\, y' + y \cos x = 0$

$$y' = \frac{\cos y + y \cos x}{x \sin y - \sin x}$$

(c) $\sin xy = \sin x + \sin y$

$(\cos xy)(y + x y') = \cos x + \cos y\, y'$

$$y' = \frac{y \cos xy - \cos x}{\cos y - x \cos xy}$$

(d) $\csc(x + y) = y$

$-\csc(x + y) \cot(x + y)(1 + y') = y'$

$$y' = -\frac{\csc(x + y) \cot(x + y)}{\csc(x + y) \cot(x + y) + 1}$$

(e) $x \tan y - y \tan x = 1$

$$\tan y + x \sec^2 y \, y' - \tan x \, y' - y \sec^2 x = 0$$

$$y' = \frac{\tan y - y \sec^2 x}{\tan x - x \sec^2 y}$$

(f) $\tan xy - x^2 = 0$

$$(\sec^2 xy)(y + x y') - 2x = 0$$

$$y' = \frac{2x - y \sec^2 xy}{x \sec^2 xy}$$

(g) $y \sin x = x \tan y$

$$\sin x \, y' + y \cos x = \tan y + x \sec^2 y \, y'$$

$$y' = \frac{\tan y - y \cos x}{\sin x - x \sec^2 y}$$

7. $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} y' = 0$

$y' = -\frac{\sqrt{y}}{\sqrt{x}}$ which is always negative since $x, y > 0$.

8. $v = \frac{4}{3} \pi r^3$

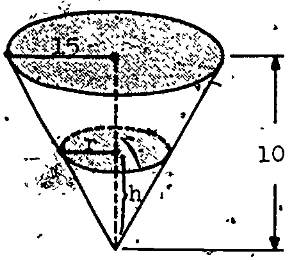
$$\frac{dv}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$$

$\frac{dv}{dt} = 100$, so when $r = 5$, $100 = \frac{4}{3} \pi \cdot 3(5^2)\frac{dr}{dt}$;

and $\frac{dr}{dt} = \frac{1}{\pi}$.

Hence, the radius is increasing $\frac{1}{\pi}$ in./min. when it is 5 inches long.

9.




$$v = \frac{1}{3} h \cdot \pi r^2$$

$$\frac{dv}{dt} = \frac{\pi}{3}\left(r^2 \frac{dh}{dt} + 2hr \frac{dr}{dt}\right)$$

From the similar triangles

$$\frac{r}{h} = \frac{15}{10}$$

so $$r = \frac{3}{2} h$$

and $$\frac{dr}{dt} = \frac{3}{2} \frac{dh}{dt} .$$

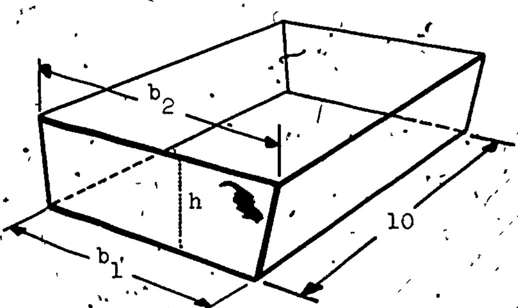
When $h = 4$, $r = 6$, and $\frac{dv}{dt} = -3$, so

$$-3 = \frac{\pi}{3}\left(6^2 \cdot \frac{dh}{dt} + 2(4)(6) \cdot \frac{3}{2} \frac{dh}{dt}\right).$$

Solving, $$\frac{dh}{dt} = -\frac{1}{12\pi}$$

and, hence, when the water level is 4 ft., it is dropping at the rate of $\frac{1}{12\pi}$ ft./min.

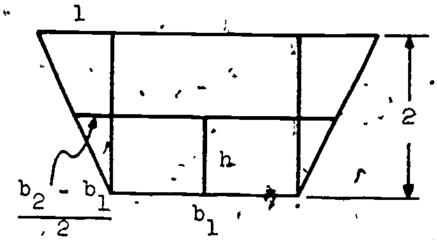
10.




$$v = 10 \cdot \frac{1}{2} h(b_1 + b_2) = 5hb_1 + 5hb_2$$

$$\frac{dv}{dt} = 5\left(h \frac{db_1}{dt} + \frac{dh}{dt} b_1\right) + 5\left(h \frac{db_2}{dt} + b_2 \frac{dh}{dt}\right)$$

By the similar triangles formed we see that




$$\frac{h}{\frac{b_2 - b_1}{2}} = \frac{2}{1}$$

$$h = b_2 - b_1$$

$$\therefore \frac{dh}{dt} = \frac{db_2}{dt} - \frac{db_1}{dt}$$

When $h = 1$, $b_1 = 1$, $b_2 = 2$, $\frac{db_1}{dt} = 0$, $\frac{db_2}{dt} = \frac{dh}{dt}$, and $\frac{dv}{dt} = 5$,

so $5 = 5(1 \cdot 0 + \frac{dh}{dt} \cdot 1) + 5(1 \cdot \frac{dh}{dt} + 2\frac{dh}{dt})$.

11. (a) $x^2 + y^2 = 2xy + 1$

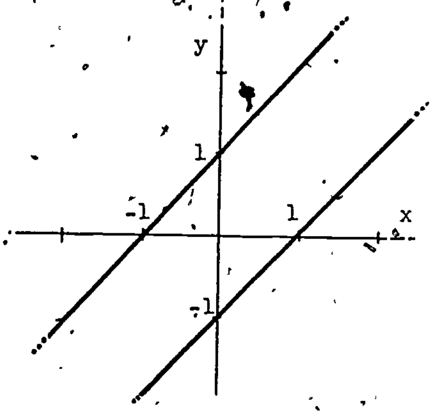
$$2x + 2y\frac{dy}{dx} = 2x\frac{dy}{dx} + 2y$$

$$2(y - x)\frac{dy}{dx} = 2(y - x)$$

$$\frac{dy}{dx} = 1, \quad y \neq x$$

If $y = x$ then $2x^2$ would equal $2x^2 + 1$, which is impossible.

(b) and (c) Graphs are the same.




$$x^2 + y^2 = 2xy + 1$$

$$x^2 - 2xy + y^2 = 1$$

$$(x - y)^2 = 1$$

$$x - y = \pm 1$$

$$|x - y| = 1$$

12.


$$s^2 = x^2 + y^2$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{s}$$

At $t = 3$, $x = 75$, $y = 300$,

$$s = \sqrt{75^2 + 300^2} \approx 309,$$

$$\frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} = 100.$$

$$\therefore \left.\frac{ds}{dt}\right|_{t=3} \approx \frac{300 \cdot 100}{309} \approx 97.$$

The rocket is receding from the observer at the approximate rate of 97 ft./sec. 3 seconds after take-off. Solving,

$$\frac{dh}{dt} = \frac{1}{4},$$

and hence, the water level is rising at the rate of $\frac{1}{4}$ ft./min.

Teachers Commentary

Chapter 9

## INTEGRATION THEORY AND TECHNIQUE

The integration techniques of Chapter 9 are extended in Appendix 4 to include the following:

- A4-1. Substitutions of Circular Functions
- A4-2. Integration by Parts
- A4-3. Integration of Rational Functions
- A4-4. Definite Integrals

### Solutions Exercises 9-1

1. (a) $\int \frac{x^2}{x^3 + a^3}\,dx;$

$$u = x^3 + a^3$$
$$du = 3x^2\,dx$$
$$\frac{1}{3}\,du = x^2\,dx$$

$$\frac{1}{3}\int u^{-1}du = \frac{1}{3}\log_e u$$
$$= \frac{1}{3}\log_e(x^3 + a^3)$$

(b) $\int x^3\sqrt[7]{1 - x^4}\,dx;$

$$u = 1 - x^4$$
$$du = -4x^3\,dx$$
$$-\frac{1}{4}\,du = x^3\,dx$$

$$-\frac{1}{4}\int u^{1/7}du = -\frac{1}{4}\,\frac{u^{8/7}}{\frac{8}{7}}$$
$$= \frac{-7}{32}(1 - x^4)^{8/7}$$

(c) $\displaystyle\int \frac{(a + b\sqrt{x})^{13}}{\sqrt{x}}\, dx,\ b \neq 0;$ $\qquad u = a + b\sqrt{x}$

$$du = \frac{b}{2\sqrt{x}}\, dx$$

$$\frac{2}{b}\, du = \frac{dx}{\sqrt{x}}$$

$$\frac{2}{b}\int u^{13}\, du = \frac{1}{7b}(a + b\sqrt{x})^{14}$$

(d) $\displaystyle\int \frac{x^2 + 1}{x - 1}\, dx;$ $\qquad u = x - 1$

$$du = dx$$

$$x = u + 1$$

$$x^2 = u^2 + 2u + 1$$

$$\int \frac{(u^2 + 2u + 1) + 1}{u}\, du = \int \left(u + 2 + \frac{2}{u}\right) du$$

$$= \frac{1}{2}\left(x^2 + 2x - 3 + 4\log_e(x - 1)\right)$$

(e) $\displaystyle\int \frac{x}{x^2 + a^2}\, dx;$ $\qquad u = x^2 + a^2$

$$du = 2x\, dx$$

$$\frac{1}{2}\, du = x\, dx$$

$$\frac{1}{2}\int u^{-1}\, du = \frac{1}{2}\log_e (x^2 + a^2) = \log_e \sqrt{x^2 + a^2}$$

(f) $\displaystyle\int \frac{x}{x^4 + a^2}\, dx,\ a \neq 0;$ $\qquad u = \frac{x^2}{a}$ and $x^2 = au$

$$du = \frac{2}{a}\, x\, dx$$

$$\frac{a}{2}\, du = x\, dx$$

$$\frac{a}{2}\int \frac{1}{a^2u^2 + a^2}\, du = \frac{a}{2}\cdot\frac{1}{a^2}\int \frac{1}{1 + u^2}\, du$$

$$= \frac{1}{2a}\arctan\frac{x^2}{a}$$

(g) $\int (\cos x) e^{\sin x} dx;$ $\quad u = \sin x$

$$du = \cos x \, dx$$

$$\int e^u \, du = e^{\sin x}$$

(h) $\int \left( \frac{ae^x}{b + ce^x} \right) dx, \; c \neq 0;$ $\quad u = b + ce^x$

$$du = ce^x \, dx$$

$$\frac{1}{c} \, du = e^x \, dx$$

$$\frac{a}{c} \int u^{-1} du = \frac{a}{c} \log_e (b + ce^x)$$

(i) $\int \sec x \, dx;$ $\quad u = \sec x + \tan x$

$$du = (\sec x \tan x + \sec^2 x) dx$$

$$du = \sec x (\tan x + \sec x) dx$$

$$\frac{du}{u} = \sec x \, dx$$

$$\int \frac{du}{u} = \log_e (\sec x + \tan x)$$

2\. (a) $\int e^{2x} dx;$ $\quad u = 2x$

$$\frac{1}{2} du = dx$$

$$\frac{1}{2} \int e^u \, du = \frac{1}{2} e^{2x}$$

(b) $\int (1 - \frac{1}{2} x)^{10} dx;$ $\quad u = 1 - \frac{1}{2} x$

$$du = -\frac{1}{2} dx$$

$$-2du = dx$$

$$-2 \int u^{10} du = -\frac{2}{11} (1 - \frac{1}{2} x)^{11}$$

(c) $\int \sin ax\ dx;$ $\quad u = ax$

$$du = a\,dx$$

$$\frac{1}{a}\,du = dx$$

$$\frac{1}{a}\int \sin u\,du = -\frac{1}{a}\cos ax$$

(d) $\int \sqrt[4]{3x + 1}\,dx;$ $\quad u = 3x + 1$

$$\frac{1}{3}\,du = dx$$

$$\frac{1}{3}\int u^{1/4}\,du = \frac{4}{15}(3x + 1)^{5/4}$$

(e) $\int \frac{1}{2 - 3x}\,dx;$ $\quad u = 2 - 3x$

$$-\frac{1}{3}\,du = dx$$

$$-\frac{1}{3}\int u^{-1}du = -\frac{1}{3}\log_e(2 - 3x)$$

(f) $\int \frac{1}{\sqrt{(1 - 5x)^3}}\,dx;$ $\quad u = 1 - 5x$

$$-\frac{1}{5}\,du = dx$$

$$-\frac{1}{5}\int u^{-3/2}du = \frac{2}{5\sqrt{1 - 5x}}$$

(g) $\int \frac{1}{a^2 + x^2}\,dx;$ $\quad u = \frac{x}{a}$ then $au = x$

$$a\,du = dx$$

$$a\int \frac{1}{a^2 + a^2u^2}\,du = \frac{1}{a}\int \frac{1}{1 + u^2}\,du$$

$$= \frac{1}{a}\arctan\frac{x}{a}$$

(h) $\int \tan(\frac{1}{2}x - 3)dx;$ $\quad u = \frac{1}{2}x - 3$

$$2du = dx$$

$$2\int \tan u\,du = -2\log_e\cos(\frac{1}{2}x - 3)$$

3. (a) $\int (4 - 3x^2)^6 \, x \, dx$;
$$u = 4 - 3x^2$$
$$du = -6x \, dx$$
$$-\frac{1}{6} du = x \, dx$$
$$-\frac{1}{6}\int u^6 \, du = -\frac{1}{42}(4 - 3x^2)^7 = \frac{1}{42}(3x^2 - 4)^7$$

(b) $\int \cos^5 x \sin x \, dx$;
$$u = \cos x$$
$$du = -\sin x \, dx$$
$$-du = \sin x \, dx$$
$$-\int u^5 \, du = -\frac{1}{6}\cos^6 x$$

(c) $\int \sin^2 2x \cos 2x \, dx$;
$$u = \sin 2x$$
$$du = 2 \cos 2x \, dx$$
$$\frac{1}{2} du = \cos 2x \, dx$$
$$\frac{1}{2}\int u^2 \, du = \frac{1}{6}\sin^3 2x$$

(d) $\int \frac{e^{1/x}}{x^2} dx$;
$$u = \frac{1}{x}$$
$$du = -\frac{1}{x^2} dx$$
$$-du = \frac{1}{x^2} dx$$
$$-\int e^u \, du = -e^{1/x}$$

(e) $\int x\sqrt{1 + 4x^2} \, dx$;
$$u = 1 + 4x^2$$
$$du = 8x \, dx$$
$$\frac{1}{8} du = x \, dx$$
$$\frac{1}{8}\int u^{1/2} du = \frac{1}{12}(1 + 4x^2)^{3/2}$$

(f) $\int \frac{(\log_e x)^2}{x}\,dx$; $\quad u = \log_e x$

$$du = \frac{1}{x}\,dx$$

$$\int u^2\,du = \frac{1}{3}(\log_e x)^3.$$

(g) $\int \frac{\cos\sqrt{2x}}{\sqrt{x}}\,dx$; $\quad u = \sqrt{2x}$

$$du = \frac{\sqrt{2}}{2\sqrt{x}}\,dx$$

$$\sqrt{2}\,du = \frac{dx}{\sqrt{x}}$$

$$\sqrt{2}\int \cos u\,du = \sqrt{2}\sin\sqrt{2x}$$

(h) $\int \frac{\sin x}{(a + b\cos x)^2}\,dx$; $\quad u = a + b\cos x$

$$du = -b\sin x$$

$$-\frac{1}{b}\,du = \sin x\,dx$$

$$-\frac{1}{b}\int u^{-2}\,du = \frac{1}{b(a + b\cos x)}$$

(i) $\int \frac{x^2}{(4x^3 - 1)^{3/2}}\,dx$; $\quad u = 4x^3 - 1$

$$du = 12x^2\,dx$$

$$\frac{1}{12}\,du = x^2\,dx$$

$$\frac{1}{12}\int u^{-3/2}\,du = \frac{-1}{6(4x^3 - 1)^{1/2}}$$

(j) $\int \frac{x}{1 + x^2}\,dx$; $\quad u = 1 + x^2$

$$du = 2x\,dx$$

$$\frac{1}{2}\,du = x\,dx$$

$$\frac{1}{2}\int u^{-1}\,du = \frac{1}{2}\log_e(1 + x^2) = \log_e\sqrt{1 + x^2}$$

(k) $\int \frac{x}{1 + x^4}\,dx;$ $\quad u = x^2$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$$\frac{1}{2}\int \frac{1}{1 + u^2}\,du = \frac{1}{2}\arctan x^2$$

($\ell$) $\int \frac{x}{\sqrt{1 - 9x^4}}\,dx;$ $\quad u = 3x^2$

$du = 6x\,dx$

$\frac{1}{6}\,du = x\,dx$

$$\frac{1}{6}\int \frac{1}{\sqrt{1 - u^2}}\,du = \frac{1}{6}\,\text{arc}\sin 3x^2$$

(m) $\int \sin^2 x \cos^3 x\,dx$: Let $\cos^3 x = \cos x(1 - \sin^2 x)$

$$\int \sin^2 x(1 - \sin^2 x)\cos x\,dx = \int (\sin^2 x - \sin^4 x)\cos x\,dx;$$

$u = \sin x$

$du = \cos x\,dx$

$$\int (u^2 - u^4)\,du = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x.$$

(n) $\int \sin^3 4x \cos^8 4x\,dx;$ $\quad \sin^3 4x = \sin 4x(1 - \cos^2 4x)$

$$\int (1 - \cos^2 4x)\cos^8 4x \sin 4x\,dx$$

$$\int (\cos^8 4x - \cos^{10} 4x)\sin 4x\,dx;$$

$u = \cos 4x$

$du = -4\sin 4x\,dx$

$-\frac{1}{4}\,du = \sin 4x\,dx$

$$-\frac{1}{4}\int (u^8 - u^{10})\,du = -\frac{1}{36}\cos^9 4x + \frac{1}{44}\cos^{11} 4x$$

4. (a) $\int_2^3 \frac{1}{(2x+1)^2}\,dx;$ $\quad u = 2x + 1$

$$du = 2\,dx$$

$$\frac{1}{2}\,du = dx$$

$$\frac{1}{2}\int_5^7 u^{-2}\,du = -\frac{1}{2}u^{-1}\Big|_5^7$$

$$= -\frac{1}{2}[\frac{1}{7} - \frac{1}{5}]$$

$$= \frac{-(5-7)}{70} = \frac{1}{35}$$

(b) $\int_0^{\pi} \cos^4 x \sin x\,dx;$ $\quad u = \cos x$

$$du = -\sin x\,dx$$

$$-du = \sin x\,dx$$

$$-\int_1^{-1} u^4\,du = -\frac{1}{5}u^5\Big|_1^{-1}$$

$$= -\frac{1}{5}[-1 - (+1)] = \frac{2}{5}$$

(c) $\int_0^{\pi/3} \cos 4x\,dx;$ $\quad u = 4x$

$$\frac{1}{4}\,du = dx$$

$$\frac{1}{4}\int_0^{4\pi/3} \cos u\,du = \frac{1}{4}\sin u\Big|_0^{4\pi/3}$$

$$= \frac{1}{4}[-\frac{\sqrt{3}}{2} - 0] = -\frac{\sqrt{3}}{8}$$

(d) $\int_{-1/2}^{0} (2x+1)^{17}\,dx;$ $\quad u = 2x + 1$

$$\frac{1}{2}\,du = dx$$

$$\frac{1}{2}\int_0^1 u^{17}\,du = \frac{1}{36}u^{18}\Big|_0^1$$

$$= \frac{1}{36}$$

(e) $\int_1^0 \sqrt{1+x}\,dx$; $u = 1 + x$

$du = dx$

$$\int_2^1 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big|_2^1$$

$$= \frac{2}{3}[1 - 2\sqrt{2}]$$

(f) $\int_{-1}^1 x\sqrt{1-x^2}\,dx$; $u = 1 - x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$$-\frac{1}{2}\int_0^0 u^{1/2}\,du = 0$$

(g) $\int_0^{1/2} \frac{1}{1+4x^2}\,dx$; $u = 2x$

$\frac{1}{2}du = dx$

$$\frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan u\Big|_0^1$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{8}$$

(h) $\int_{-1/2}^{1/2} \frac{1}{\sqrt{9-x^2}}\,dx$; $u = \frac{x}{3}$ thus $3u = x$

$3\,du = dx$

$$3\int_{-1/6}^{1/6} \frac{1}{\sqrt{9-9u^2}}\,du = \frac{3}{3}\int_{-1/6}^{1/6} \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \arcsin u\Big|_{-1/6}^{1/6}$$

(arc sin .166$\bar{6}$ ≈ 0.167 from Table 3.)

$$\approx 0.167 - (-0.167)$$

$$\approx 0.334$$

(i) $\int_1^2 \frac{2 \log_e x}{x}\,dx$; $\quad u = \log_e x$

$$du = \frac{1}{x}\,dx$$

$$\int_{\log_e 1}^{\log_e 2} u\,du = \frac{1}{2}u^2 \Big|_0^{\log_e 2}$$

$$= \frac{1}{2}(\log_e 2)^2 \approx .24$$

(j) $\int_0^1 x^3 \sqrt{1 - x^2}\,dx$; $\quad u = 1 - x^2, \quad x^2 = 1 - u$

$$du = -2x\,dx$$

$$-\frac{1}{2}\,du = x\,dx$$

$$-\frac{1}{2}\int_1^0 (1 - u)u^{1/2}\,du = -\frac{1}{2}\int_1^0 (u^{1/2} - u^{3/2})\,du$$

$$= -\frac{1}{2}\left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_1^0$$

$$= -\frac{1}{2}\left[0 - \left(\frac{2}{3} - \frac{2}{5}\right)\right] = \frac{2}{15}$$

(k) $\int_{-1}^1 x^2 e^{x^3}\,dx$; $\quad u = x^3$

$$du = 3x^2\,dx$$

$$\frac{1}{3}\,du = x^2\,dx$$

$$\frac{1}{3}\int_{-1}^1 e^u\,du = \frac{1}{3}e^u \Big|_{-1}^1$$

$$= \frac{1}{3}\left[e - \frac{1}{e}\right]$$

($\ell$) $\int_0^{\sqrt{\pi/2}} x \sin(2x^2)\,dx$; $\quad u = 2x^2$

$$du = 4x\,dx$$

$$\frac{1}{4}\,du = x\,dx$$

$$\frac{1}{4}\int_0^{\pi} \sin u\,du = -\frac{1}{4}\cos u \Big|_0^{\pi} = -\frac{1}{4}[-1 - (+1)]$$

$$= \frac{1}{2}$$

5. (a) $\int \frac{\cos x}{\sqrt{9 - \sin^2 x}}\, dx$; $\quad u = \sin x$

$du = \cos x\, dx$

$\int \frac{du}{\sqrt{9 - u^2}}$; $\quad v = \frac{u}{3}$ or $3v = u$

$3dv = du$

$$3\int \frac{dv}{\sqrt{9 - 9v^2}} = 3\int \frac{dv}{3\sqrt{1 - v^2}}$$

$$= \frac{3}{3} \text{ arc sin } v$$

$$= \text{arc sin } \frac{u}{3}$$

$$= \text{arc sin } (\tfrac{1}{3} \sin x)$$

(b) $\int \frac{x^2}{2 + x^6}\, dx$; $\quad u = x^3$

$du = 3x^2\, dx$

$\frac{1}{3}\, du = x^2\, dx$

$\frac{1}{3}\int \frac{1}{2 + u^2}\, du$; $\quad v = \frac{u}{\sqrt{2}}$

$\sqrt{2}\, v = u$

$\sqrt{2}\, dv = du$

$$\frac{\sqrt{2}}{3}\int \frac{1}{2(1 + v^2)}\, dv = \frac{\sqrt{2}}{6} \text{ arc tan } v$$

$$= \frac{\sqrt{2}}{6} \text{ arc tan } \frac{u}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{6} \text{ arc tan } \frac{x^3}{\sqrt{2}}$$

(c) $\int \frac{1}{\sqrt{x} + x}\,dx;$ $u = \sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}\,dx$$

$$2u\,du = dx$$

$$2\int \frac{u\,du}{u + u^2} = 2\int \frac{du}{1 + u}; \qquad v = 1 + u$$

$$dv = du$$

$$2\int \frac{1}{v}\,dv = 2\log_e v$$

$$= 2\log_e (1 + u)$$

$$= 2\log_e (1 + \sqrt{x})$$

6. (a) $\int x^2 \sin (x - 1)dx;$ $u = (x - 1)$

$$u + 1 = x$$

$$du = dx$$

$$\int (u^2 + 2u + 1)\sin u\,du = \int u^2 \sin u\,du + 2\int u \sin u\,du + \int \sin u\,du$$

(Integrals No. 21, No. 20, No. 4 and No. 22)

$$= (-u^2 - 2u + 1)\cos u + 2(u + 1)\sin u$$

$$= (1 - x^2)\cos(x - 1) + 2x \sin (x - 1)$$

(b) $\int_0^2 x\,e^{2x}\,dx;$ $u = 2x$

$$\frac{1}{2}\,du = dx$$

$$\frac{1}{2}u = x$$

$$\frac{1}{4}\int_0^4 u\,e^u\,du \quad \text{(Integral No. 16)}$$

$$= \frac{1}{4}(ue^u - e^u)\Big|_0^4$$

$$= \frac{1}{4}(3e^4 + 1)$$

(c) $\int_0^{\pi} x \sin 3x\, dx;$ $\quad u = 3x$

$$\frac{1}{3}\, du = dx$$

$$\frac{1}{3}\, u = x$$

$$\frac{1}{9} \int_0^{3\pi} u \sin u\, du \quad \text{(Integral No. 20)}$$

$$= \frac{1}{9}(-u \cos u + \sin u) \Big|_0^{3\pi}$$

$$= \frac{\pi}{3}$$

(d) $\int x \cos^3 (x^2)dx;$ $\quad u = x^2$

$$du = 2x\, dx$$

$$\frac{1}{2}\, du = x\, dx$$

$$\frac{1}{2} \int \cos^3 u\, du \quad \text{(Integral No. 29)}$$

$$= \frac{\cos^2 u \sin u}{3} + \frac{2}{3} \sin u$$

$$= \frac{\cos^2 x^2 \sin x^2}{3} + \frac{2}{3} \sin x^2$$

(e) $\int x^3 e^{-4x};$ $\quad u = -4x$

$$x^3 = -\frac{u^3}{64}$$

$$-\frac{1}{4}\, du = dx$$

$$\frac{1}{256} \int u^3 e^u\, du \quad \text{(Integral No. 17)}$$

$$= \frac{e^u}{256}[u^3 - 3u^2 + 6u - 6]$$

$$= \frac{-e^{-4x}}{128}[32x^3 + 24x^2 + 12x + 3]$$

(f) $\int xe^{x^2} \sin 2x^2\, dx$;  $u = 2x^2$

$$x^2 = \frac{u}{2}$$

$$du = 4x\, dx$$

$$\frac{1}{4} \int e^{u/2} \sin u\, du \quad \text{(Integral No. 24)}$$

$$= \frac{1}{5} e^{u/2}\left(\frac{1}{2} \sin u - \cos u\right)$$

$$= \frac{1}{10} e^{x^2} \left(\sin 2x^2 - 2 \cos 2x^2\right)$$

(g) $\int_0^1 x^2 \log_e (x + 1)dx$;  $u = x + 1$

$$x^2 = (u - 1)^2$$

$$du = dx$$

$$\int_1^2 (u^2 - 2u + 1)\log_e u\, du \quad \text{(Integral No. 19, No. 18 and No. 7)}$$

$$= \left(\frac{u^3}{3} - u^2 + u\right)\log_e u + \left(-\frac{u^3}{9} + \frac{u^2}{2} - u\right)\Big|_1^2$$

$$= \frac{2}{3} \log_e 2 - \frac{5}{18}$$

(h) $\int \sin x \log_e (\cos x)dx$;  $u = \cos x$

$$du = -\sin x\, dx$$

$$-\int \log u\, du \quad \text{(integral No. 7)}$$

$$= -u \log_e u + u$$

$$= -\cos x \log_e \cos x + \cos x$$

(i) $\int \sin(x + 1)\cos(2x + 2)\,dx;$ $\quad u = x + 1$

$du = dx$

$\int \sin u \cos 2u \, du$ (Integral No. 32)

$$= \frac{\cos(-u)}{2} + \frac{\cos 3u}{6}$$

$$= \frac{\cos(-(x + 1))}{2} + \frac{\cos(3(x + 1))}{6}$$

$$\text{or } \frac{\cos(x + 1)}{2} + \frac{\cos(3(x + 1))}{6}$$

(j) $\int \frac{\sin x}{2 \cos^2 x + \cos x - 3}\,dx;$ $\quad u = \cos x$

$du = -\sin x \, dx$

$-\int \frac{1}{2u^2 + u - 3}\,du;$ $\quad b^2 - 4ac > 0$

(Integral No. 36)

$$= \frac{1}{5} \log_e \frac{2u - 2}{2u + 3}$$

$$= \frac{1}{5} \log_e \frac{2 \cos x - 2}{2 \cos + 3}$$

(k) $\int \frac{e^x}{4e^{2x} - 2e^x + 1}\,dx;$ $\quad u = e^x$

$du = e^x dx$

$\int \frac{1}{4u^2 - 2u + 1}\,du;$ $\quad b^2 - 4ac < 0$

(Integral No. 37)

$$= \frac{1}{\sqrt{3}} \arctan \frac{4u - 1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \arctan \frac{4e^x - 1}{\sqrt{3}}$$

(ℓ) $\int \frac{1}{x\sqrt{(\log_e x)^2 + 1}}\, dx;$ $\quad u = \log_e x$

$du = \frac{1}{x}\, dx$

$\int \frac{1}{\sqrt{u^2 + 1}}\, du;$ (Integral No. 38)

$= \log_e (u + \sqrt{u^2 + 1})$

$= \log_e(\log_e x + \sqrt{(\log_e x)^2 + 1})$

7. $\int_0^1 e^{-x^2}\, dx \approx \alpha$

(a) The function $e^{-x^2}$ is an even function since $f(x) = f(-x)$.

Thus $\int_{-1}^{0} e^{-x^2}\, dx = \int_0^1 e^{-x^2}\, dx = \alpha$

(b) $\int_{-1}^{1} e^{-x^2}\, dx = \int_{-1}^{0} e^{-x^2}\, dx + \int_0^1 e^{-x^2}\, dx$

$= \alpha + \alpha = 2\alpha \ldots$

(c) $\int_{-1}^{3} e^{-\frac{(x - 1)^2}{4}}\, dx;$ $\quad u = \frac{x - 1}{2}$

$du = \frac{1}{2}\, dx$

$2\, du = dx$

$2 \int_{-1}^{1} e^{-u^2}\, du = 2(2\alpha) = 4\alpha$

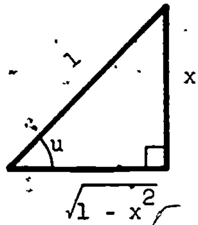
(d) $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\, dx;$ $\quad u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\, dx$

$2\, du = \frac{1}{\sqrt{x}}\, dx$

$2 \int_0^1 e^{-u^2}\, du = 2\alpha$

8. (a) $\int_0^1 \sqrt{1 - x^2}\, dx$; $u = \arcsin x$




Then $x = \sin u$

$dx = \cos u \cdot du$

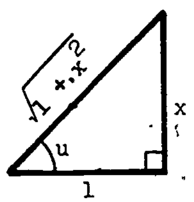
and $\sqrt{1 - x^2} = \cos u$

when $x = 0$ $u = 0$

when $x = 1$ $u = \frac{\pi}{2}$

$$\int_0^{\pi/2} \cos u \cos u \, du = \frac{u}{2} + \sin 2u \Big|_0^{\pi/2} = \frac{\pi}{4}$$

(b) $\int \frac{\sqrt{1 + x^2}}{x^4}\, dx$; $u = \arctan x$




Then $x = \tan u$

$dx = \sec^2 u \, du$

and $\sqrt{1 + x^2} = \sec u$

$$\int \frac{\sec u \cdot \sec^2 u}{\tan^4 u}\, du = \int \frac{\cos u}{\sin^4 u}\, du$$

$v = \sin u$

$dv = \cos u \, du$

$$\int v^{-4}\, dv = \frac{v^{-3}}{-3}$$

$$= -\frac{1}{3} \csc^3 u$$

$$= -\frac{1}{3} \csc^3 \arctan x$$

$$= -\frac{1}{3} \frac{(1 + x^2)^{3/2}}{x^3}$$

9. (a) If $x < 0$, then $-x = t > 0$ and $dx = -dt$, so

$$\int \frac{1}{x}\,dx = \int \left(-\frac{1}{t}\right)(-dt) = \int \frac{1}{t}\,dt = \log t$$

since $t > 0$.

$$\therefore \int \frac{1}{x}\,dx = \log(-x) = \log|x| \quad \text{since } x < 0.$$

(b) If $a < 0 < b$, we would want

$$\int_a^b \frac{1}{x}\,dx = \int_a^0 \frac{1}{x}\,dx + \int_0^b \frac{1}{x}\,dx,$$

but these two integrals are infinite if indeed they can be said to exist at all (see Exercise 7-6, No. 37). If we persisted, we would arrive at the meaningless nonsense that

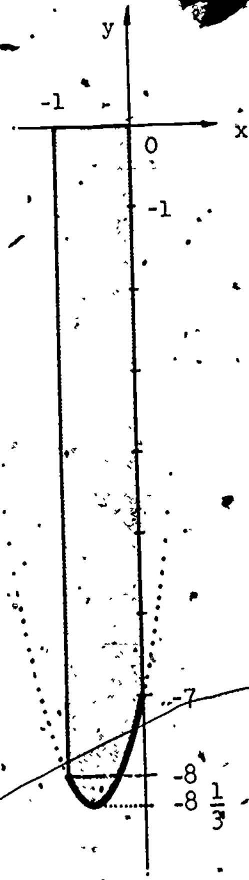
$$\int_a^b \frac{1}{x}\,dx = \log|0| - \log|a| + \log|b| - \log|0|,$$

meaningless because $\log 0$ is undefined. We conclude that the integration interval $[a,b]$ must not contain zero.

Solutions Exercises 9-2

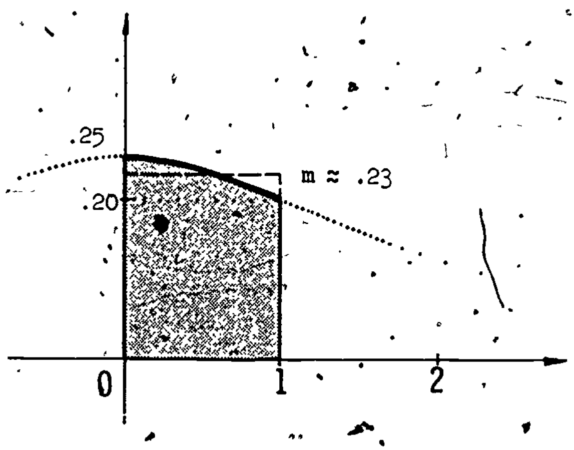
1. (a) $f : x \to 3x^2 + 4x - 7, \quad -1 \leq x \leq 0$

$$m = \frac{1}{0 - (-1)} \int_{-1}^{0} (3x^2 + 4x - 7) = -8$$




(b) $f : x \to \frac{1}{4 + x^2}, \quad 0 \leq x \leq 1$

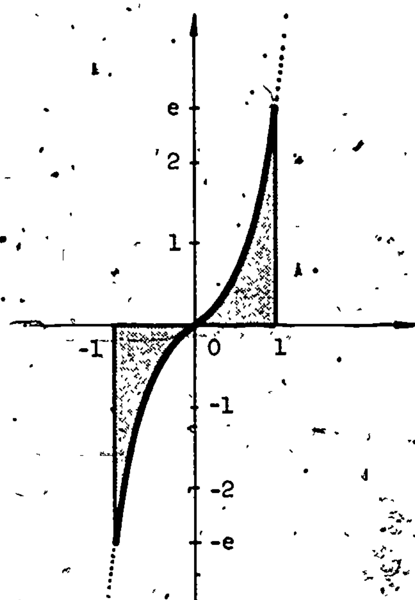
$$m = \frac{1}{1 - 0} \int_{0}^{1} \frac{1}{4 + x^2}\, dx \approx .23$$



(c) $f : s \to se^{s^2}, \quad -1 \le s \le 1$

$$m = \frac{1}{1 - (-1)} \int_{-1}^{1} s\, e^{s^2}\, ds$$

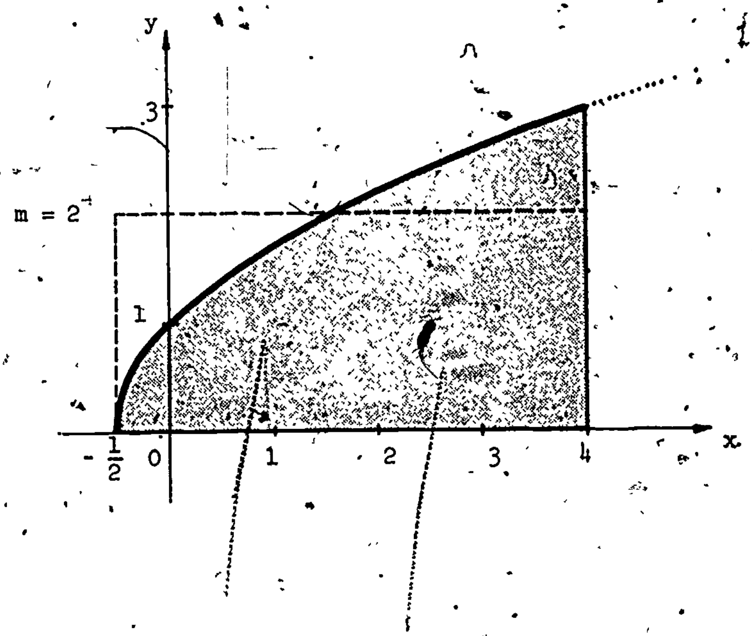
$$= \frac{1}{2}(0)$$




(d) $f : t \to \sqrt{2t + 1}, \quad -\frac{1}{2} \le t \le 4$

$$m = \frac{1}{4 - (-\frac{1}{2})} \int_{-1/2}^{4} \sqrt{2t - 1}\, dt$$

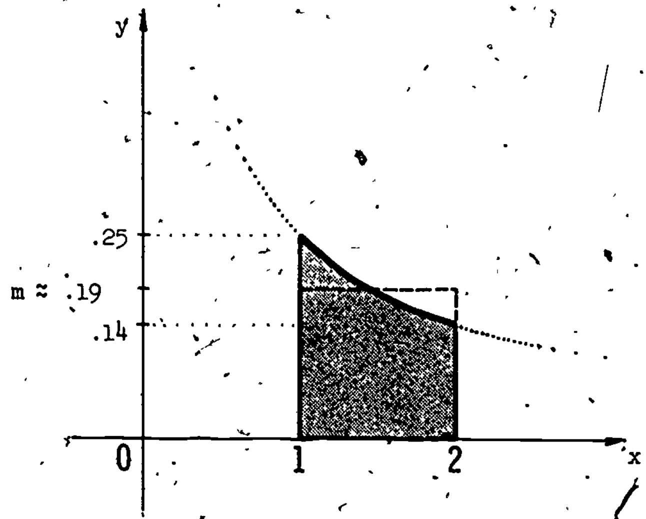
$$= \frac{2}{9}(9) = 2$$



(e) $f : x \to \frac{1}{3x + 1}, \quad 1 \le x \le 2$

$$m = \frac{1}{2 - 1} \int_1^2 \frac{1}{3x + 1}\,dx$$

$$\approx .19$$




2. $f : x \to \sin x$

(a) $0 \le x \le \pi$

$$m = \frac{1}{\pi} \int_0^\pi \sin x\,dx$$

$$= \frac{2}{\pi}$$

(b) $1 + 7\pi \le x \le 1 + 9\pi$

$$m = \frac{1}{(9\pi + 1) - (7\pi + 1)} \int_{1+7\pi}^{1+9\pi} \sin x\,dx$$

$$= 0$$

(c) $-\pi \le x \le \pi$

$$m = \frac{1}{\pi - (-\pi)} \int_{-\pi}^{\pi} \sin x\,dx$$

$$= 0$$

(d) $c \leq x \leq c + 2\pi$

$$m = \frac{1}{(c + 2\pi) - c} \int_{c}^{c+2\pi} \sin x \, dx$$

$$= 0$$

3. If $f$ is periodic with period $\alpha$ then $f(x) = f(x + n\alpha)$, $n = 0, \pm 1, \pm 2, \ldots,$ for all $x$.

Let $m$ be the average value for the interval $[x, x + \alpha]$

$$m = \frac{1}{(x + \alpha) - x} \int_{x}^{x+\alpha} f = \frac{1}{\alpha} [F(x + \alpha) - F(x)] \Big|_{x}^{x+\alpha}$$

where $F'(x) = f(x)$ and $F'(x + \alpha) = f(x + \alpha)$. This is true for all values of $x$, even for a special value of $x$ such as $x = 0$. Thus

$$m = \frac{1}{\alpha} [F(\alpha) - F(0)]$$

which is a constant.

4. The average value of the slope of the tangent is simply $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

In this case where $f : x \to x^2 + 1$ then for the interval $-1 \leq x \leq 3$,

$$m = \frac{10 - 2}{3 - (-1)} = 2.$$

Alternate solution:

The slope of $f : x \to x^2 + 1$ is $f'(x) = 2x$. The average value of the slope is

$$m = \frac{1}{3 - (-1)} \int_{-1}^{3} 2x \, dx$$

$$= \frac{1}{4} (8)$$

$$= 2$$

5. $f : x \to x^2$

$$f_{av} = \frac{1}{1-0}\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\therefore\ (f_{av})^2 = \frac{1}{9}$$

$$f^2 : x \to x^4$$

$$(f^2)_{av} = \frac{1}{1-0}\int_0^1 x^4\,dx = \frac{x^5}{5}\Big|_0^1 = \frac{1}{5}$$

$$\therefore\ (f_{av})^2 \neq (f^2)_{av}$$

6. The average acceleration is

$$m = \frac{1}{4-1}\int_1^4 (t^3 + \frac{1}{\sqrt{t}})dt$$

$$= \frac{1}{3}\cdot\frac{263}{4} = \frac{263}{12}.$$

7. Let $f : x \to ax + b$ be a linear function. The average value of $f$ on the interval $p \le x \le q$ is

$$m = \frac{1}{q-p}\int_p^q (ax+b)dx$$

$$= \frac{1}{q-p}[\frac{ax^2}{2} + bx]\Big|_p^q$$

$$= \frac{1}{q-p}[\frac{a}{2}q^2 + bq - \frac{a}{2}p^2 - bp]$$

$$= \frac{1}{(q-p)}[\frac{a}{2}(q^2-p^2) + b(q-p)]$$

$$= \frac{a}{2}(q+p) + b$$

$$= (\frac{a}{2}p + \frac{b}{2}) + (\frac{a}{2}q + \frac{b}{2})$$

$$= \frac{f(p)+f(q)}{2}$$

Solutions Exercises 9-3

1. (a) Cross-sectional area at $x$ is $\pi(3x)^2$.

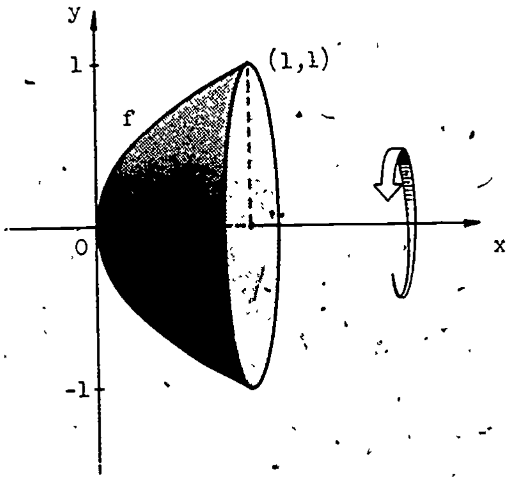
Average cross-sectional area $= \frac{1}{2-0}\int_0^2 \pi(3x)^2\,dx$

$\therefore$ Volume = (length) $\times$ (average cross sectional area)

$$= 2 \cdot \frac{1}{2}\int_0^2 \pi(3x)^2\,dx$$

$$= 9\pi\int_0^2 x^2\,dx = 9\pi \cdot \frac{x^3}{3}\Big|_0^2 = 24\pi$$

(b)

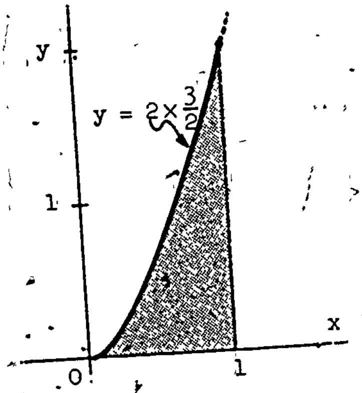



Cross-sectional area at $x$ is $\pi(\sqrt{x})^2 = \pi x$

Average cross-sectional area $= \frac{1}{1-0}\int_0^1 \pi x\,dx$

$\therefore$ Volume $= (1-0)\cdot\frac{1}{(1-0)}\int_0^1 \pi x\,dx = \pi\,\frac{x^2}{2}\Big|_0^1 = \frac{\pi}{2}$

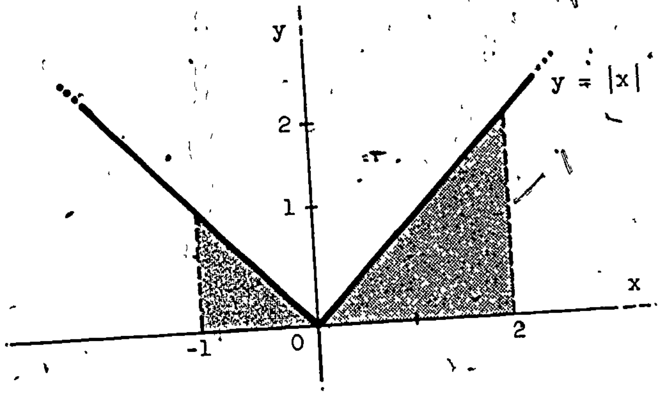
(c)




Average cross-sectional area $= \frac{1}{1-0}\int_0^1 \pi(2x^{3/2})^2\,dx$

$\therefore$ Volume $= (1-0)\cdot\frac{1}{1-0}\int_0^1 4\pi\, x^3\,dx = 4\pi\,\frac{x^4}{4}\Big|_0^1 = \pi$

(d)




For $-1 \le x < 0$, $y = |x| = -x$, so $f : x \to -x$.

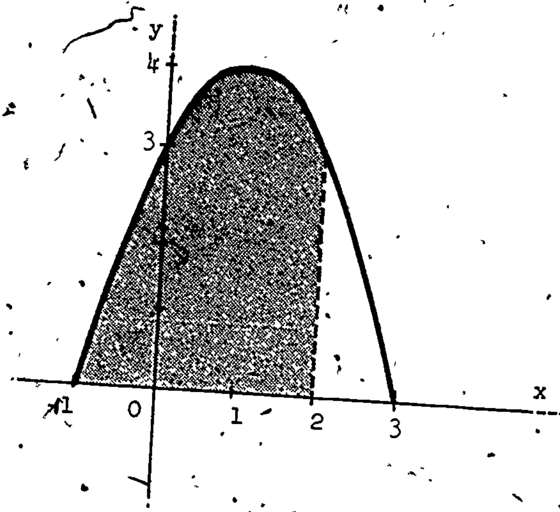
For $0 \le x \le 2$, $y = |x| = x$, so $f : x \to x$.

The total volume is the sum of the volumes of the two solids of revolution generated by these functions on the indicated intervals.

$\therefore$ Volume $= \int_{-1}^{0} \pi(-x)^2\,dx + \int_0^2 \pi(x)^2\,dx$

$= \pi \int_{-1}^{2} x^2\,dx = \pi\,\frac{x^3}{3}\Big|_{-1}^{2} = \pi\,[\frac{8}{3} - (\frac{-1}{3})] = 3\pi$

(e)

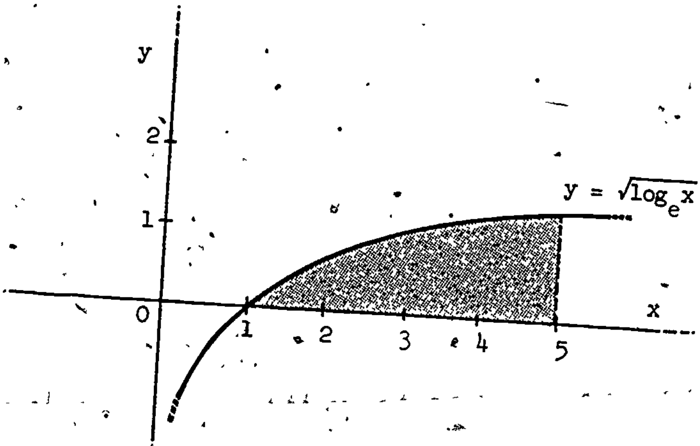



$$\text{Volume} = \int_{-1}^{2} \pi[-(x-1)^2+4]^2 dx = \pi \int_{-1}^{2} [(x-1)^4 - 8(x-1)^2 + 16]dx$$

$$= \pi\left[\frac{(x-1)^5}{5} - \frac{8(x-1)^3}{3} + 16x\right]\Big|_{-1}^{2} = \pi\left(\frac{1}{5} - \frac{8}{3} + 32 + \frac{32}{5} - \frac{64}{3} + 16\right)$$

$$= \frac{153\pi}{5}$$

(f)

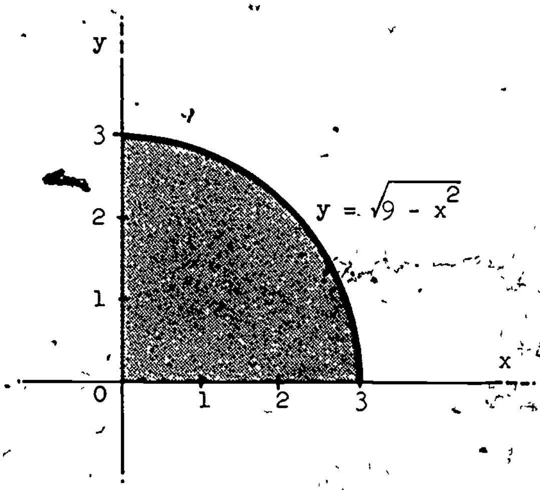



$$\text{Volume} = \int_{1}^{5} \pi(\sqrt{\log_e x})^2 \, dx = \pi \int_{1}^{5} \log_e x \, dx$$

$$= \pi(x \log_e x - x)\Big|_{1}^{5} = \pi(5 \log_e 5 - 5 - \log_e 1 + 1)$$

$$= \pi(5 \log_e 5 - 4) \approx 4.047\pi \approx 12.71$$

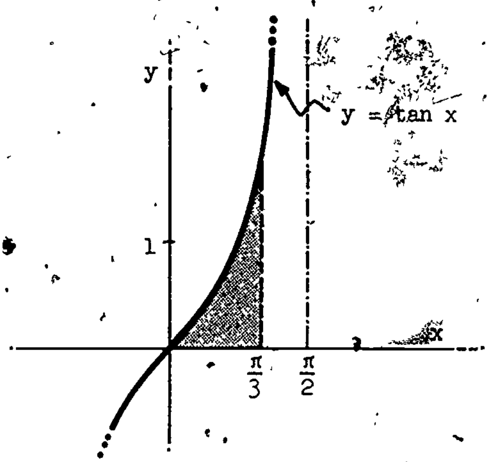
(g)




$$\text{Volume} = \int_0^3 \pi(\sqrt{9 - x^2})^2 dx = \pi \int_0^3 (9 - x^2)dx$$

$$= \pi(9x - \frac{x^3}{3}) \Big|_0^3 = \pi(27 - \frac{27}{3}) = 18\pi$$

(h)



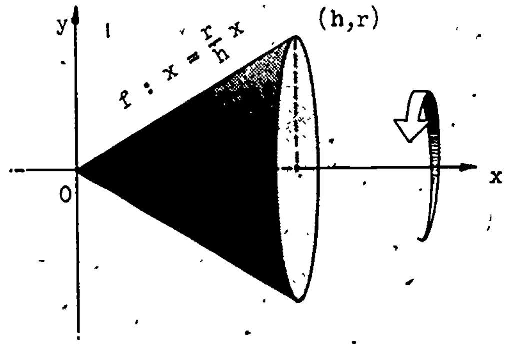

$$\text{Volume} = \int_0^{\pi/3} \pi(\tan x)^2 \, dx = \pi \int_0^{\pi/3} \tan^2 x \, dx = \pi \int_0^{\pi/3} (\sec^2 x + 1)dx$$

$$= \pi(\tan x + x) \Big|_0^{\pi/3} \pi(\sqrt{3} - \frac{\pi}{3}).$$

2. $V = (h - 0)\,\frac{1}{h - 0}\int_0^h \pi \cdot \left(\frac{r}{h}\,x\right)^2 dx$

$= h \cdot \frac{1}{h}\,\pi\,\frac{r^2}{h^2}\,\frac{x^3}{3}\Big|_0^h$

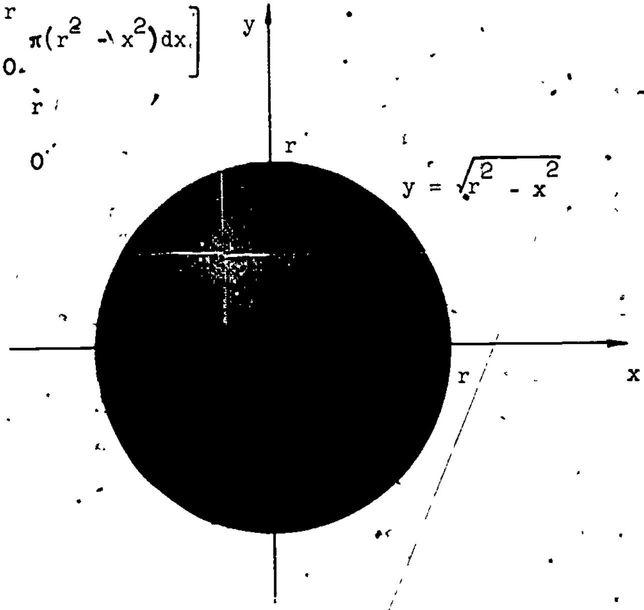
$= \frac{\pi r^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3}\pi\, r^2 h$




3. $V = 2\left[(r - 0) \cdot \frac{1}{(r - 0)}\int_0^r \pi(r^2 - x^2)\,dx\right]$

$= 2 \cdot r \cdot \frac{1}{r} \cdot \pi\left(r^2 x - \frac{x^3}{3}\right)\Big|_0^r$

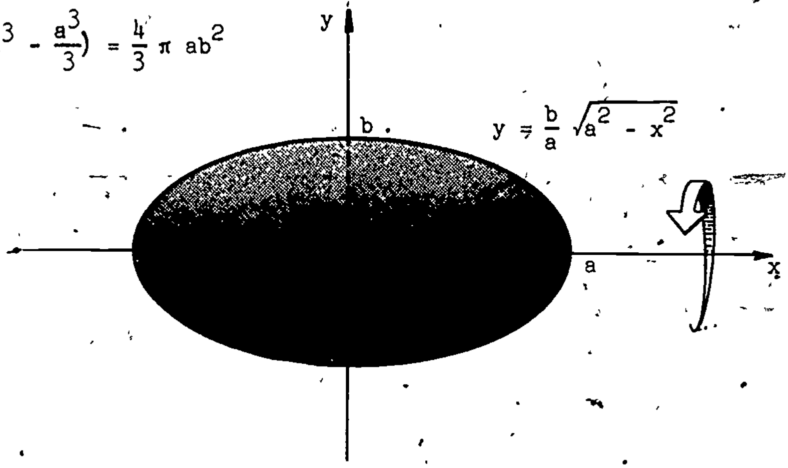
$= 2\pi\left(r^3 - \frac{r^3}{3}\right) = \frac{4}{3}\pi r^3$



4. $V = 2\left[(a - 0) \cdot \frac{1}{(a - 0)} \int_0^a \pi \frac{b^2}{a^2} (a^2 - x^2) dx\right]$

$= 2 \cdot a \cdot \frac{1}{a} \cdot \pi \cdot \frac{b^2}{a^2} (a^2x - \frac{x^3}{3}) \Big|_0^a$

$= \frac{2\pi b^2}{a^2} \cdot (a^3 - \frac{a^3}{3}) = \frac{4}{3} \pi ab^2$

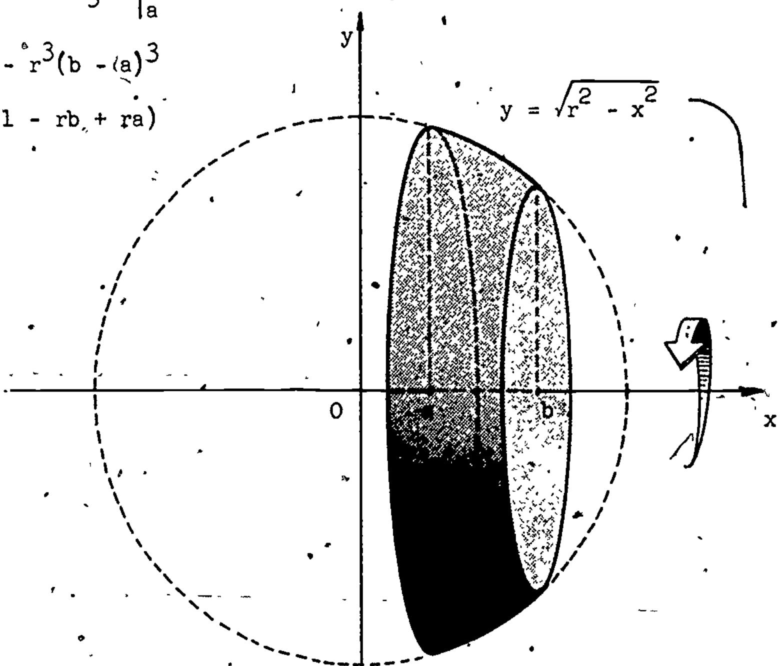



5. $V = b - a \cdot \frac{1}{b - a} \int_a^b \pi(r^2 - x^2) dx$

$= \frac{(b - a)}{b - a} \pi(r^2x - \frac{r^3}{3}) \Big|_a^b$

$= r^2(b - a) - r^3(b - a)^3$

$= r^2(b - a)(1 - rb + ra)$



6. $V = V_1 - V_2$

where $f_1 : x = 2\sqrt{x}$

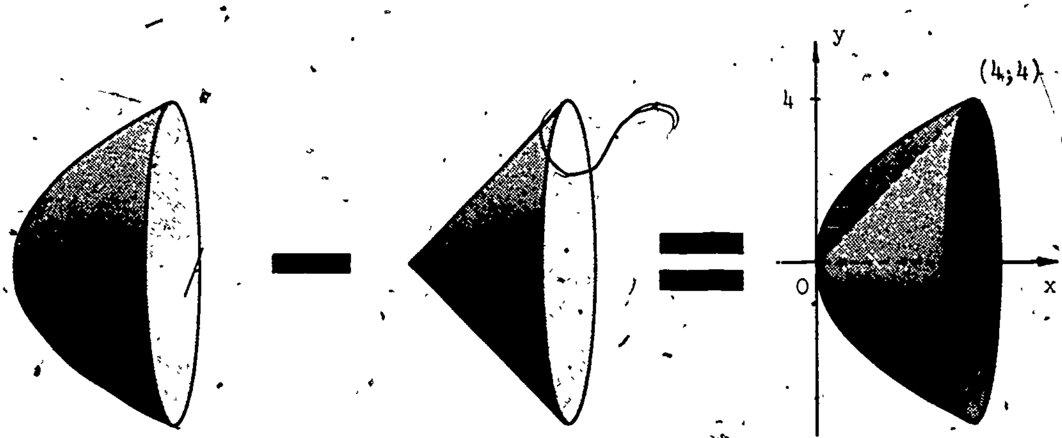
$f_2 : x = x$

$$V = 4 - 0 \cdot \frac{1}{4-0}\int_0^4 \pi(4x)\,dx - 4 - 0\,\frac{1}{4-0}\int_0^4 \pi(x^2)\,dx$$

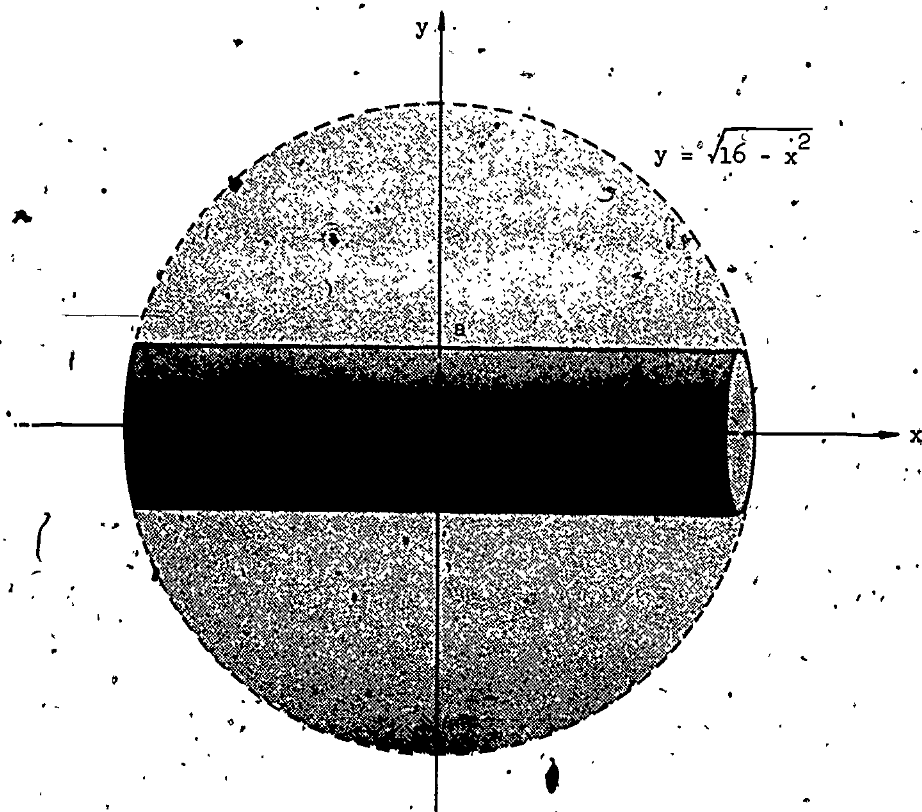
$$= \frac{4}{4}\cdot\pi\int_0^4 (4x - x^2)\,dx$$

$$= \pi\left(2x^2 - \frac{x^3}{3}\right)\Big|_0^4$$

$$= \pi\left(32 - \frac{6x}{3}\right) = \frac{32}{3}\pi$$



7. The required volume consists of a cylinder of 1 inch in radius and two "caps" each formed like a shallow dome.




The value of $x$ for which $y = 1$ is $x = \sqrt{15}$. Thus the length of the cylinder is $2\sqrt{15}$ and its volume is $V_{cy} = \pi \cdot 1^2 \cdot 2\sqrt{15} = 2\pi\sqrt{15}$.

The volume of the 2 "caps" is found by

$$2V_{cap} = 2\left[(4 - \sqrt{15})\frac{1}{(4 - \sqrt{15})}\int_{\sqrt{15}}^{4}\pi(16 - x^2)dx\right]$$

$$= 2\pi(16x - \frac{x^3}{3})\Big|_{\sqrt{15}}^{4}$$

$$= 2\pi[(64 - \frac{64}{3}) - (16\sqrt{15} - 5\sqrt{15})]$$

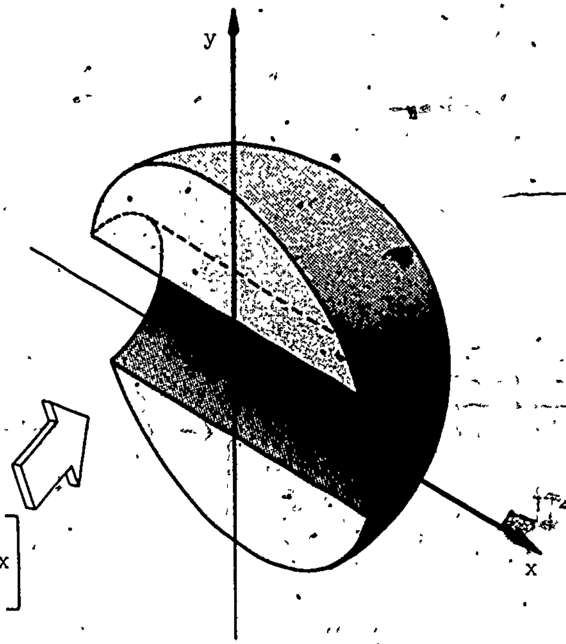
$$= \pi[\frac{256}{3} - 22\sqrt{15}]$$

$$2V_{cap} + V_{cy} = \pi[\frac{256}{3} - 22\sqrt{15} + 2\sqrt{15}] \approx \pi[85\frac{1}{3} - 20\sqrt{15}]$$

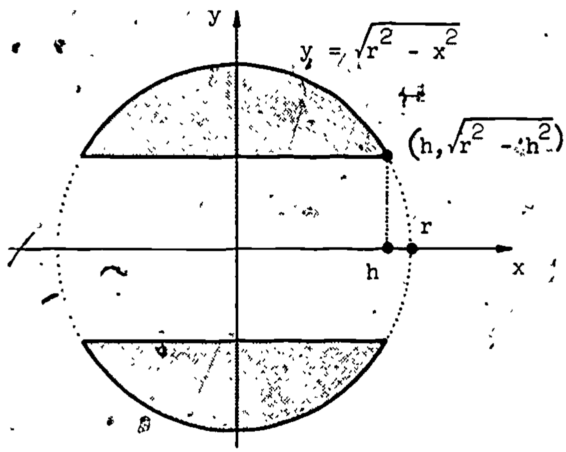
$$\approx \pi(7.87366) \approx 24.7$$

It is interesting to note that the volume of the cylinder is $\approx 24.3^{+}$. Very little is contributed by the caps.

8. The cross-sectional area will be the difference of the areas of two circles. The larger will have a radius $y = \sqrt{r^2 - x^2}$, the smaller will have a constant radius of $\sqrt{r^2 - h^2}$. The cross-sectional area is given by



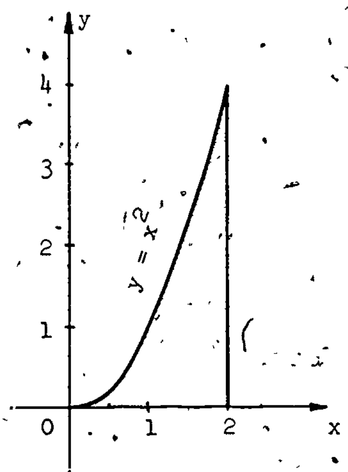
$$C(x) = \pi((r^2 - x^2) - (r^2 - h^2))$$
$$= \pi(h^2 - x^2)$$
$$V = 2\left[(h - 0)\frac{1}{h - 0}\int_0^h \pi(h^2 - x^2)\,dx\right]$$
$$= 2\pi\left(h^2 x - \frac{x^3}{3}\right)\Big|_0^h$$
$$= 2\pi\,\frac{2h^3}{3} = \frac{4}{3}\pi h^3$$




Special values such as $h = 0$ and $h = r$ yield

$$V_{h=0} = 0 \text{ and } V_{h=r} = \frac{4\pi}{3} r^3.$$

9. The volumes desired in this problem can be accomplished by translations of the shaded area in the figure. The lines about which we wish to revolve will be the axes in the revolved system.



(a) Let us define the translation as $T : (x,y) \to (x,y + 4)$

Then $T : (y = x^2) \to y + 4 = x^2$

or $y = x^2 - 4$.

The cross-section is

$$C(x) = \pi[4^2 - (x^2 - 4)^2]$$

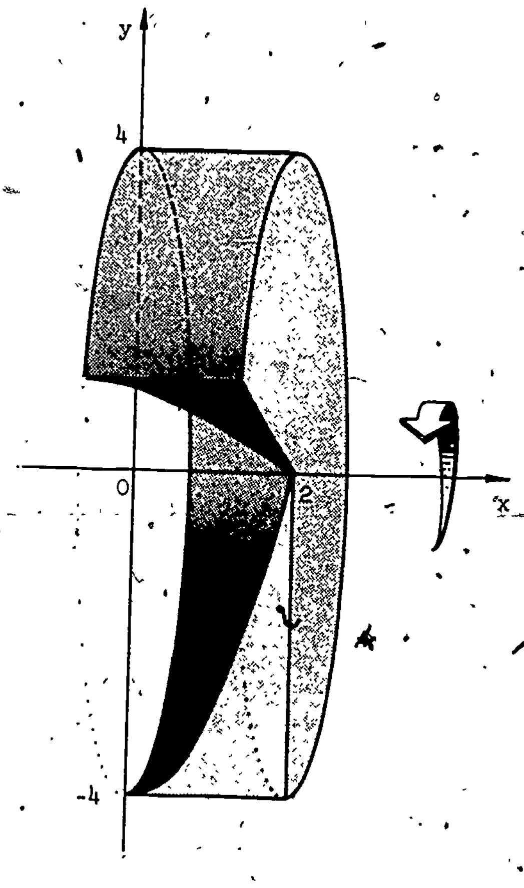
$$= \pi(-x^4 + 8x^2).$$

Thus $V$ = length × average $C(x)$

$$= (2 - 0) \cdot \frac{1}{2 - 0} \int_0^2 \pi(-x^4 + 8x^2)dx$$

$$= 2\,\frac{1}{2} \cdot \pi(-\frac{x^5}{5} + \frac{8x^3}{3}) \Big|_0^2$$

$$= \pi(-\frac{32}{5} + \frac{64}{3}) = \frac{224}{15}\pi$$



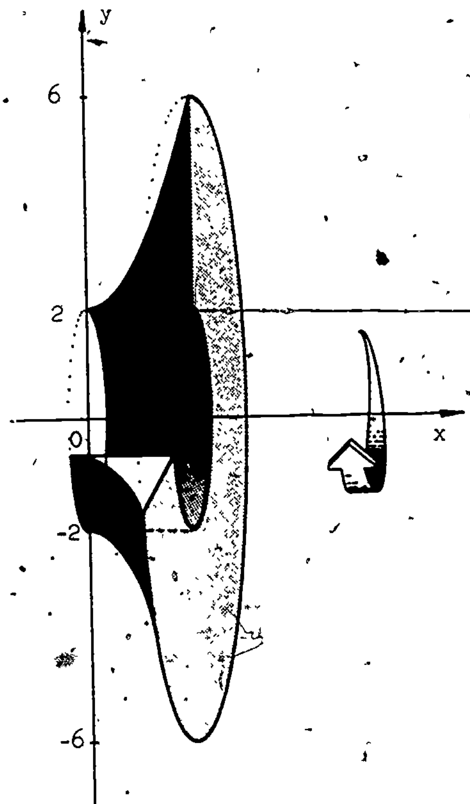
(b) Let the transformation be $T : (x,y) \to (x,y - 2)$

Thus $T : (y = x^2) \to (y - 2 = x^2)$ or $y = x^2 + 2$.

The average cross-section is

$$C(x) = \pi((x^2 + 2)^2 - 2^2)$$
$$= \pi(x^4 + 4x^2)$$

$$V = (2 - 0)\frac{1}{2 - 0}\int_0^2 \pi(x^4 + 4x^2)dx$$
$$= 2 \cdot \frac{\pi}{2}\left(\frac{x^5}{5} + \frac{4x^3}{3}\right)\Big|_0^2$$
$$= \pi\left(\frac{32}{5} + \frac{32}{3}\right) = \frac{256}{15}\pi$$



(c) Let $T : (x,y) \rightarrow (x + 2,y)$

$$y = (x + 2)^2$$

or $$x = \sqrt{y} - 2$$

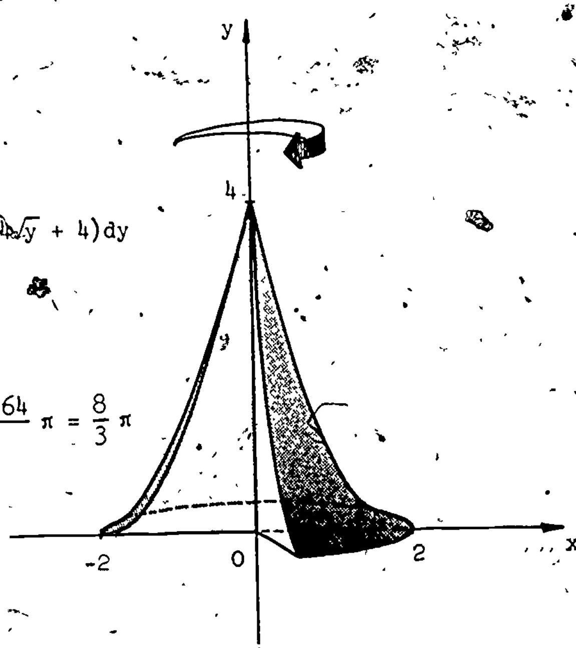
$$C(x) = \pi(\sqrt{y} - 2)^2$$

$$= \pi(y - 4\sqrt{y} + 4)$$

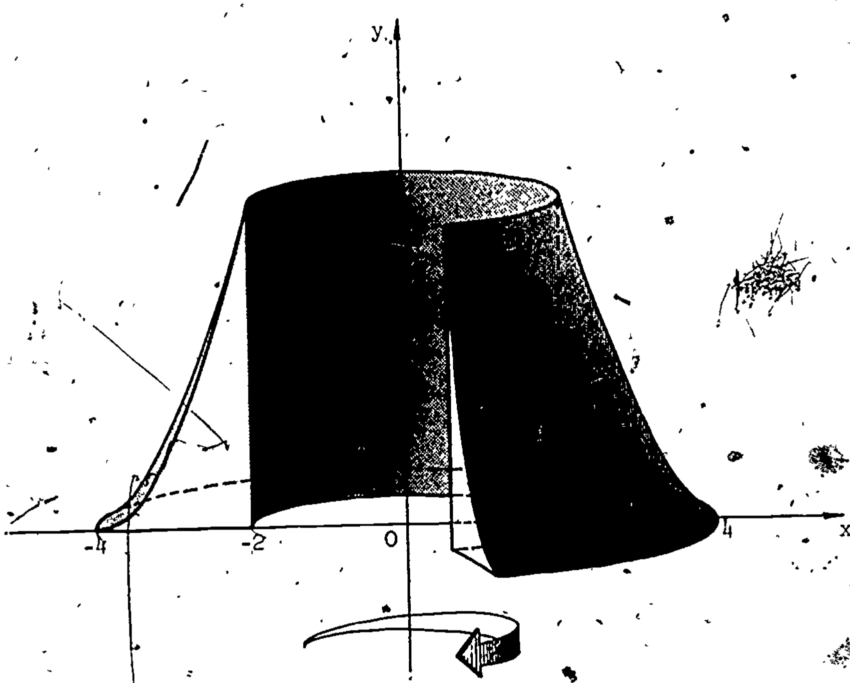
$$V = 4 - 0 \cdot \frac{1}{4 - 0}\int_0^4 \pi(y - 4\sqrt{y} + 4)dy$$

$$= \pi\left(\frac{y^2}{2} - \frac{4y^{3/2}}{\frac{3}{2}} + 4y\right)\Bigg|_0^4$$

$$= \pi\left(8 - \frac{64}{3} - 16\right) = \frac{72 - 64}{3}\pi = \frac{8}{3}\pi$$




Let the translation to $T : (x,y) \rightarrow (x + 4,y)$. Then $T : (y = x^2) \rightarrow (y = (x + 4)^2)$ and $x = \sqrt{y} - 4$



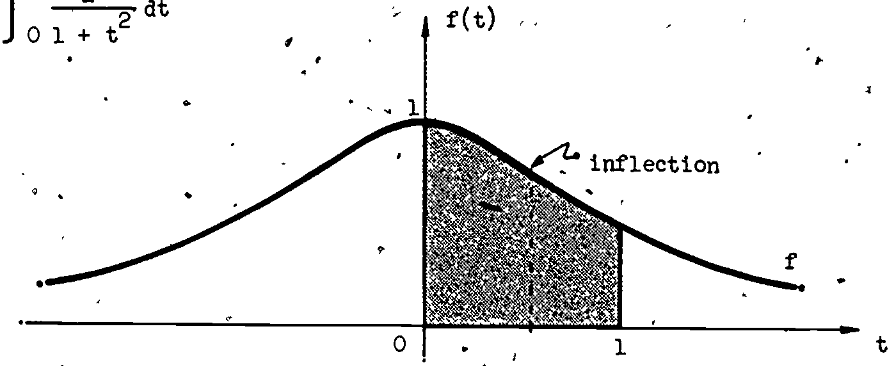
The cross-sectional area is

$$C(x) = \pi((\sqrt{y} - 4)^2 - 2^2)$$
$$= \pi(y - 8\sqrt{y} + 12)$$

$$V = (4 - 0)\frac{1}{4 - 0}\int_0^4 \pi(y - 8\sqrt{y} + 12)dy$$
$$= 4 \cdot \frac{\pi}{4}\left(\frac{y^2}{2} - \frac{16y^{3/2}}{3} + 12y\right)\Bigg|_0^4$$
$$= \pi\left(8 - \frac{128}{3} + 48\right) = \frac{40}{3}\pi$$

## Solutions Exercises 9-4

1. $\int_0^1 \frac{1}{1 + t^2} dt$




(a) If $n = 2$ $t_1 = 0, t_1 = \frac{1}{2}, t_2 = 1$

| t | f(t) |
|---|---|
| 0 | 1 |
| $\frac{1}{2}$ | $\frac{4}{5}$ |
| 1 | $\frac{1}{2}$ |

$$\int_0^1 f \approx \frac{1 - 0}{2 \cdot 2} (f(0) + 2f(\tfrac{1}{2}) + f(1))$$

$$\approx \frac{1}{4}(1 + 2(\tfrac{4}{5}) + \tfrac{1}{2}) \approx \frac{31}{40} \approx 0.775$$

$$f'(t) = \frac{-2t}{(1 + t^2)^2}$$

$$f''(t) = \frac{2(3t^2 - 1)}{(1 + t^2)^3}$$

On the interval $[0,1]$, $f''(t)$ is maximum when $t = 0$. Then $|f''(0)| = |-2|$. Let $M = 1$ and the error is at most $\frac{2(1 - 0)^3}{12(2)^2} = \frac{1}{24} \approx .041\overline{6}$.

(b) If $n = 4$ then $t_0 = 0$, $t_1 = \frac{1}{4}$, $t_2 = \frac{1}{2}$, $t_3 = \frac{3}{4}$ and $t_4 = 1$.

| $t$ | $f(t)$ |
|---|---|
| 0 | 1 |
| $\frac{1}{4}$ | $\frac{16}{17} \approx 0.94$ |
| $\frac{1}{2}$ | $\frac{4}{5} = 0.80$ |
| $\frac{3}{4}$ | $\frac{16}{25} = 0.64$ |
| 1 | $\frac{1}{2}$ |

$$\int_0^1 f(t)dt \approx \frac{1 - 0}{2 \cdot 4}(1 + 1.88 + 1.60 + 1.28 + 0.5)$$

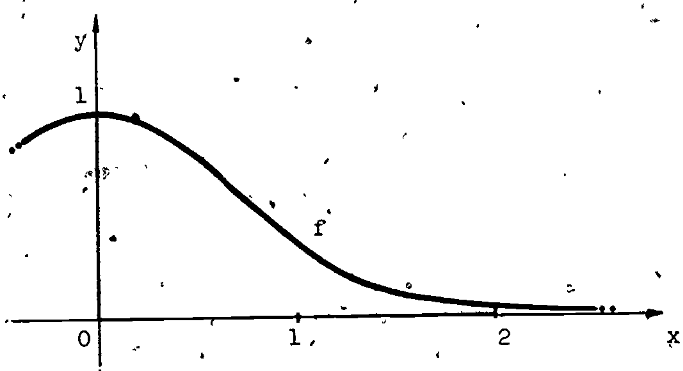
$$\approx \frac{3.13}{4} \approx 0.783$$

When $M = 1$ the maximum error is $\frac{2(1 - 0)^3}{12(4^2)} = \frac{1}{96} \approx 0.0104$.

The exact value of $\int_0^1 \frac{1}{1 + t^2} dt$ is found by integration.

$$\int_0^1 \frac{1}{1 + t^2} dt = \arctan t \Big|_0^1 = \frac{\pi}{4} \approx .785398 \approx .785$$

2. $\int_0^2 e^{-x^2} dx$




(a)

| $y$ | $f$ |
|---|---|
| 0 | 1.000 |
| $\frac{1}{2}$ | 0.7788 |
| 1 | 0.3679 |
| $\frac{3}{2}$ | 0.1054 |
| 2 | 0.0183 |

$$n = 2 \quad \int_0^2 e^{-x^2} dx \approx \frac{2 - 0}{2 \cdot 2}(1 + 0.7358 + 0.0183)$$

$$\approx \frac{1.7541}{2} \approx 0.8771$$

(b) $n = 4$ $\int_0^2 e^{-x^2}\,dx \approx \frac{2-0}{2 \cdot 4}(1 + 1.5576 + 0.7358 + 0.2108 + 0.0183)$

$$\approx \frac{3.5225}{4} \approx 0.8806 .$$

To determine the maximum error first find $f'$ and $f''$.

$$f' = -2xe^{-x^2} \text{ and } f'' = 4x^2e^{-x^2}$$

$|f''|$ is maximum when $x = 2$.

$$|f''(2)| = |16e^{-4}| \approx 16(0.0183) \approx 0.2928$$

Let $M = 0.2928$, then the maximum error is $\frac{0.2928(2-0)^3}{12 \cdot 2^2} \approx 0.0488$ when $n = 2$, or $\frac{0.2928(2-0)^3}{12 \cdot 4^2} \approx 0.0122$ when $n = 4$.

3\. $\int_0^1 \frac{1}{1+t^2}\,dt$

(a) $n = 1$ Use the values of Problem 1.

$$\int_0^1 \frac{1}{1+t^2}\,dt \approx \frac{1-0}{6 \cdot 1}\left(1 + 4\left(\frac{4}{5}\right) + \frac{1}{2}\right)$$

$$\approx \frac{4.7}{6} \approx .78\overline{3}$$

This is the same as the actual value correct to two places.

(b) $n = 3$ Six such intervals are needed.

| $t$ | $f(t)$ | multiple of $f(t)$ | |
|---|---|---|---|
| $t_0 = 0$ | $1$ | 1 | 1.000 |
| $t_1 = \frac{1}{6}$ | $\frac{36}{37}$ | 4 | 3.892 |
| $t_2 = \frac{2}{6}$ | $\frac{36}{40} = \frac{9}{10}$ | 2 | 1.800 |
| $t_3 = \frac{3}{6}$ | $\frac{36}{45} = \frac{4}{5}$ | 4 | 3.200 |
| $t_4 = \frac{4}{6}$ | $\frac{36}{52} = \frac{9}{13}$ | 2 | 1.385 |
| $t_5 = \frac{5}{6}$ | $\frac{36}{61}$ | 4 | 2.361 |
| $t_6 = \frac{6}{6}$ | $\frac{36}{72}$ | 1 | .500 |
| | | sum | 14.138 |

$$\int_0^1 \frac{1}{1+t^2}\,dt \approx \frac{1-0}{6\cdot 3}(14.138)$$

$$\approx .785445 \approx .7854$$

This is the same as the actual value correct to four places. (Refer to Problem 1.)

The estimate of error requires finding the third and fourth derivatives.

$$f'(t) = \frac{-2t}{(1+t^2)^2}$$

$$f''(t) = \frac{2(3t^2-1)}{(1+t^2)^3}$$

$$f'''(t) = \frac{24t(1-t^2)}{(1+t^2)^4}$$

$$f^{(4)}(t) = \frac{12(t^4-12t^2+2)}{1+t^2}$$

$$f^4(1) = 54 = M$$

Then the error $\leq \dfrac{54(1-0)^5}{180(2\cdot 1)^4} \approx .03750$ for $n = 1$, and

the error $\leq \dfrac{54(1-0)^5}{180(2\cdot 3)^4} \approx 0.0002315$ for $n = 3$.

4. $\int_0^2 e^{-x^2}\,dx$

| $x$ | $e^{-x^2}$ | $n = 1$ | $n = 2$ |
|---|---|---|---|
| 0 | 1 | $f(1) = 1.0000$ | $f(1) = 1.0000$ |
| $\frac{1}{2}$ | 0.7788 | | $4f(\frac{1}{2}) = 3.1152$ |
| 1 | 0.3679 | $4f(1) = 1.4716$ | $2f(1) = .7358$ |
| $\frac{3}{2}$ | 0.1054 | | $4f(\frac{3}{2}) = .4216$ |
| 2 | 0.0183 | $f(2) = 0.0366$ | $f(2) = 0.0366$ |
| | Sum | 2.5082 | 5.3092 |

(a) $\int_0^2 e^{-x^2}\,dx \approx \frac{2-0}{6\cdot 1}(2.5082)$ when $n = 1$

$\approx .83606 \approx .84$

(b) $\int_0^2 e^{-x^2}\,dx \approx \frac{2-0}{6\cdot 2}(5.3092)$ when $n = 2$

$\approx .884868 \approx .885$

To estimate the error we first find $M$.

$$f' = -2xe^{-x^2}$$

$$f'' = 2e^{-x^2}(2x^2 + 1)$$

$$f''' = 16x^3 e^{-x^2}$$

$$f^{(4)} = 16x^2 e^{-x^2}(2x^2 - 3)$$

$f^{(4)}$ is its greatest at $x = \frac{1}{2}$ on the interval $0 \le x \le 2$.

[Note: $x = \frac{1}{2}$ is a zero of $f^{(5)}$.]

$$f^{(4)}(0) = 0$$

$$f^{(4)}(\tfrac{1}{2}) \approx 7.788$$

$$f^{(4)}(1) \approx 5.886$$

$$f^{(4)}(\tfrac{3}{2}) \approx 5.692$$

$$f^{(4)}(2) \approx 1.171$$

Let $M = 8$ then the error is less than

$$\frac{8(2-0)^5}{180(2\cdot 1)^4} \approx .0\overline{8} \quad \text{when } n = 1.$$

and less than

$$\frac{8(2-1)^5}{180(2\cdot 2)^4} \approx .00\overline{5} \quad \text{when } n = 2.$$

5. Let $f : x \to Ax^3 + Bx^2 + Cx + D$ where $B = C = D = 0$.

The area $\int_a^b Ax^3 = \frac{A}{4}(b^4 - a^4)$ by actual integration.

By Simpsons Rule

$$\int_a^b Ax^3 = \frac{b-a}{6}\left[f(a) + 4f\frac{(a+b)}{2} + f(b)\right]$$

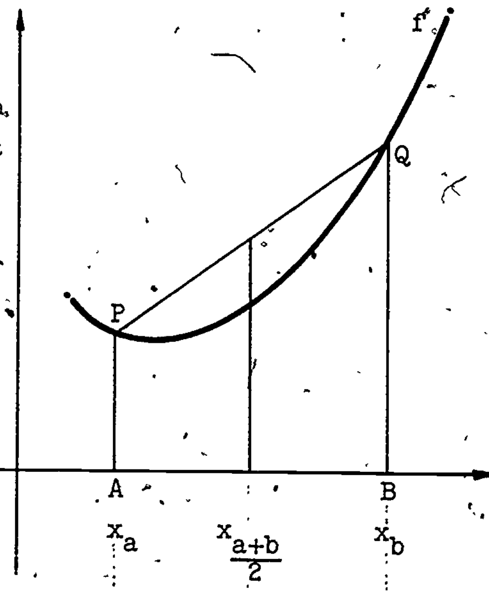
$$= \frac{A(b-a)}{6}\left[a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right]$$

$$= \frac{A(b-a)}{6}\left[a^3 + \frac{1}{2}(b^3 + 3b^2a + 3ba^2 + a^3) + b^3\right]$$

$$= \frac{A}{4}(b-a)(b^3 + b^2a + ba^2 + b^3)$$

$$= \frac{A}{4}(b^4 - a^4)$$

6. If $f$ is convex then each point of $f$ between $a$ and $b$ is below each point on the interior of the segment $\overline{PQ}$. Thus the area of the trapezoid ABQP must be greater than the area found by $\int_a^b f$.




7. In the case of the Trapezoid Rule (3) the number $n$ is the number of subintervals into which $[a,b]$ is partitioned. Whereas in the case of Simpsons Rule (8) the interval $[a,b]$ is actually partitioned into $2n$ subintervals.

In (3) $n$ represents the number of trapezoidal regions and in (8) $n$ represents the number of parabolic regions into which $f$ is divided over the domain $[a,b]$.

8. From problem 3 we found that $54 \le M$. In order to insist on accuracy to five decimal places, $n$ must satisfy this inequality.

$$\frac{54(1-0)^5}{180(2n)^4} \le .000005$$

Solving for an integer value of $n$,

$$\frac{3}{160n^4} \le \frac{1}{2 \times 10^5}$$

$$\frac{3 \times 10^4}{8} \le n^4$$

$$61.24 \le n^2$$

and

$$8 \le n$$ will do quite well.

9. $\log_e 3 = \int_1^3 \frac{1}{x}\,dx$

(i) Trapezoidal Rule:

If we are to insure four decimal place accuracy then

$$\frac{M(3-1)^3}{12n^2} \le \frac{1}{2 \times 10^4}$$

$$f : x \to \frac{1}{x}$$

$$f' : x \to -\frac{1}{x^2}$$

$$f'' : x \to 2\frac{1}{x^3} .$$

Let $2\left(\frac{1}{1}\right) \le M$.

$$\frac{2(2)^3}{12n^2} \le \frac{1}{2 \times 10^4}$$

$$\frac{2^5 \times 10^4}{12} \le n^2$$

$$163 < n$$

We had better choose a different method!

(ii) Simpsons Rule:

$$f''' = -6\frac{1}{x^4}$$

$$f^{(4)} = 24\frac{1}{x^5}$$

$$M = 24$$

The required value of $n$ must satisfy this inequality

$$\frac{24(3-1)^5}{180(2n)^4} \le \frac{1}{2 \times 10^4}$$

$$\frac{24 \cdot 2^5 \cdot 2 \cdot 10^4}{180 \cdot 2^4} \le n^4$$

$$\frac{24 \cdot 2 \cdot 10^3}{18\ 9\ 3}$$

$$\frac{16 \cdot 10^3}{3} \le n^4$$

$$5,333 \le n^4$$

$$73 \le n^2$$

$$9 \le n$$

We can let $n = 10$. This means that we must partition the interval $[1,3]$ into $2n = 20$ subintervals.

| $x$ | $\frac{1}{x}$ | Multiple of $\frac{1}{x}$ | |
|---|---|---|---|
| 1.0 | 1.00000 | 1 | 1.00000 |
| 1.1 | .90909 | 4 | 3.63636 |
| 1.2 | .83333 | 2 | 1.66667 |
| 1.3 | .76923 | 4 | 3.07692 |
| 1.4 | .71428 | 2 | 1.42856 |
| 1.5 | .66667 | 4 | 2.66667 |
| 1.6 | .62500 | 2 | 1.25000 |
| 1.7 | .58824 | 4 | 2.35296 |
| 1.8 | .55556 | 2 | 1.11111 |
| 1.9 | .52632 | 4 | 2.10528 |
| 2.0 | .50000 | 2 | 1.00000 |
| 2.1 | .47619 | 4 | 1.90476 |
| 2.2 | .45455 | 2 | .90909 |
| 2.3 | .43478 | 4 | 1.73912 |
| 2.4 | .41667 | 2 | .83333 |
| 2.5 | .40000 | 4 | 1.60000 |
| 2.6 | .38462 | 2 | .76923 |
| 2.7 | .37037 | 4 | 1.48148 |
| 2.8 | .35714 | 2 | .71428 |
| 2.9 | .34483 | 4 | 1.37932 |
| 3.0 | .33333 | 1 | .33333 |
| | | | 32.95847 |
| | | | SUM |

$$\int_1^3 \frac{1}{x}\,dx \approx \frac{(3-1)}{6 \cdot 10}(32.95847)$$

$$\approx 1.09860$$

The actual value of $\log_e 3$ is 1.09861. Our answer turns out to be correct to 4 places as directed.

1. $-1 \leq \cos x \leq 1$

(a) $\int_0^x -dt \leq \int_0^x \cos t\, dt \leq \int_0^x dt$

$-x \leq \sin x \leq x$

(b) $\int_0^x -t\, dt \leq \int_0^x \sin t\, dt \leq \int_0^x t\, dt$

$-\frac{x^2}{2} \leq 1 - \cos x \leq \frac{x^2}{2}$

(c) $\int_0^x -\frac{t^2}{2}\, dt \leq \int_0^x (1 - \cos t)dt \leq \int_0^x \frac{t^2}{2}\, dt$

$-\frac{x^3}{3!} \leq x - \sin x \leq \frac{x^3}{3!}$

(d) $\int_0^x -\frac{t^3}{3!}\, dt \leq \int_0^x (t - \sin t)dt \leq \int_0^x \frac{t^3}{3!}\, dt$

$-\frac{x^4}{4!} \leq \cos x - (1 - \frac{x^2}{2}) \leq \frac{x^4}{4!}$

(e) $\int_0^x -\frac{t^4}{4!} \leq \int_0^x (\cos t - 1 + \frac{t^2}{2})dt \leq \int_0^x \frac{t^4}{4!}\, dt$

$-\frac{x^5}{5!} \leq \sin x - (x - \frac{x^3}{3!}) \leq \frac{x^5}{5!}$

2. If $x \leq 0$ we will change each result depending on whether it is an even or odd function. If $f$ is even then $f(x) = f(-x)$, whereas if $f$ is odd then $f(x) = -f(-x)$.

(a) $-(-x) \leq \sin(-x) \leq (-x)$

$x \leq -\sin x \leq -x, \quad x \leq 0$

(b) $-\frac{x^2}{2} \leq 1 - \cos x \leq \frac{x^2}{2}$, for all $x$. These are even functions thus $f(x) = f(-x)$.

(c) $-\frac{(-x)^3}{3!} \leq (-x) - \sin(-x) \leq \frac{(-x)^3}{3!}$

$\frac{x^3}{3!} \leq -x + \sin x \leq -\frac{x^3}{3!}, \quad x \leq 0.$

(d) $-\frac{x^4}{4!} \le \cos x - (1 - \frac{x^2}{2!}) \le \frac{x^4}{4!}$, for all $x$.

(e) $-\frac{(-x)^5}{5!} \le \sin(-x) - ((-x) - \frac{(-x)^3}{3!}) \le \frac{(-x)^5}{5!}$

$$\frac{x^5}{5!} \le -\sin x + (x + \frac{x^3}{3!}) \le -\frac{x^5}{5!}, \ x \le 0$$

3. $f : x \to \sqrt[3]{1 + x}$ $\qquad f(0) = 1$

$f' : x \to \frac{1}{3}(1 + x)^{-2/3}$ $\qquad f'(0) = \frac{1}{3}$

$f'' : x \to -\frac{1}{3} \cdot \frac{2}{3}(1 + x)^{-5/3}$ $\qquad f''(0) = -\frac{2}{9}$

$f''' : x \to \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3}(1 + x)^{-8/3}$ $\qquad f'''(0) = \frac{10}{27}$

$f^{(4)} : x \to -\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} \cdot \frac{8}{3}(1 + x)^{-11/3}$ $\qquad f^{(4)}(0) = -\frac{80}{81}$

$$p_3(x) = 1 + \frac{1}{3}x - \frac{2}{9} \cdot \frac{x^2}{2!} + \frac{10}{27} \cdot \frac{x^3}{3!}$$
$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$$

$f(x) = p_3(x) + R_3$

On the interval $[0,1]$, $|f^{(4)}(x)|$ is a maximum when $x = 0$ and $|f^4(0)| = \frac{80}{81} \le K$. From (16), $|R_3| \le K \frac{|x|^4}{4!} \le K \frac{M^4}{4!}$, $0 \le x \le 1$. Let $x = 1$ since this is the maximum value of $x$ on the interval $[0,1]$. Then

$$|R_3| \le \frac{80}{81} \cdot \frac{1}{4!} = \frac{10}{243} \approx .04115.$$

4. (a) $f : x \to \sqrt{1 + x}$, $x$ on the interval $-1 < x \le 0$

$f^4 : x \to -\frac{15}{16}(1 + x)^{-7/2}$

$|f^{(4)}(x)| \le K$

$K$ is maximum when $x \to -1$.

$$\lim_{x \to -1} f^{(4)}(x) = \infty$$

Thus $|R_n| \le K \frac{|x|^{n+1}}{(n + 1)!}$

$$|R_3| \le \lim_{x \to -1} f^4(x) \frac{(1)^4}{4!} \to \infty$$

There is no error estimate possible near $x = -1$.

(b) A more realistic problem is to estimate error over a closed interval. We selected $-.5 \le x \le 0$.

$$f^{(4)}(-.5) = -\frac{15}{16} \cdot 2^{7/2} \qquad f^{4}(0) = -\frac{15}{16}$$

$$|f^{(4)}(x)| \le \frac{15}{2}\sqrt{2}$$

Thus $|R_3| \le \frac{15}{2}\sqrt{2}\,\frac{|-.5|^4}{4!} = \frac{5\sqrt{2}}{256} \approx .02762.$

5. (a) $f : x \to \frac{1}{1+x}$ $\qquad f(0) = 1$

$f' : x \to \frac{-1}{(1+x)^2}$ $\qquad f'(0) = -1$

$f'' : x \to \frac{2!}{(1+x)^3}$ $\qquad f''(0) = 2!$

$\vdots \qquad \vdots$

$f^{(n-1)} : x \to \frac{(n-1)!}{(1+x)^n}$ $\qquad f^{(n-1)}(0) = (-1)^{n-1}(n-1)!$

$$p_{n-1}(x) = 1 - x + \frac{2!}{2!}x^2 - \frac{3!}{3!}x^3 + \dots (-1)^{(n-1)}\frac{(n-1)!}{(n-1)!}x^{n-1}$$

$$= 1 - x + x^2 - x^3 + \dots (-1)^{n-1}x^{n-1} + R_{n-1}$$

(b) If $|R_{n-1}| \le \frac{|x|^n}{|1+x|}$ then $|R_n| \le \frac{|x|^{n+1}}{|x+1|}$

(c) For $p_5(10)$, $R_n \le \frac{10^6}{11} \approx 909{,}091$.

This is very large as should be expected.

(d) $\lim\limits_{n\to\infty} \frac{1}{1+x} \to 0$ whereas $\lim\limits_{n\to\infty} |R_n| \to \infty$.

When $x > 1$. Since $p_n(x)$ is an alternating series, $p_n(x)$ oscillates wildly as $n \to \infty$.

6. $f : x \to \frac{1}{2 + x} = \frac{1}{2}\left(\frac{1}{1 + \frac{x}{2}}\right)$ $\qquad f(0) = \frac{1}{2}$

$f' : x \to \frac{1}{2} \cdot \frac{1}{2} \frac{-1}{(1 + \frac{x}{2})^2}$ $\qquad f'(0) = (\frac{1}{2})^2$

$f'' : x \to \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2!}{(1 + \frac{x}{2})^3}$ $\qquad f''(0) = (\frac{1}{2})^3\ 2!$

$f''' : x \to \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{-3!}{(1 + \frac{x}{2})^4}$ $\qquad f'''(0) = -(\frac{1}{2})^4\ 3!$

$\vdots \qquad\qquad\qquad\qquad \vdots$

$f^{(n-1)} : x \to (\frac{1}{2})^{n-1}(-1)^n \frac{n!}{(1 + \frac{x}{2})^{n+1}}$ $\qquad f^{(n-1)}(0) = (-1)^n(\frac{1}{2})^{n-1} n!$

$$p_{n-1}(x) = \frac{1}{2} - (\frac{1}{2})^2 x + (\frac{1}{2})^3 x^2 - (\frac{1}{2})^4 x^3 \ldots (-1)^n(\frac{1}{2})^{n-1} x^{n-1}$$

$$= \frac{1}{2}(1 - \frac{x}{2} + (\frac{x}{2})^2 - (\frac{x}{2})^3 + \ldots (-1)^n(\frac{x}{2})^{n-1})$$

$$R_{n-1} = \frac{(\frac{x}{2})^n}{2(1 + \frac{x}{2})} = \frac{(\frac{x}{2})^n}{(2 + x)}$$

As $n \to \infty$ the only values of $x$ for which $R_{n-1} \to 0$ is when

$|\frac{x}{2}| < 1$ or $|x| < 2$.

7. $f : x \to \log_e(2 + x)$ $\qquad f(0) = \log_e 2$

$f' : x \to \frac{1}{2 + x} = \frac{1}{2(1 + \frac{x}{2})}$ $\qquad f'(0) = \frac{1}{2}$

$f'' : x \to \frac{1}{2} \cdot \frac{1}{2} \frac{-1}{(1 + \frac{x}{2})^2}$ $\qquad f''(0) = -(\frac{1}{2})^2$

$f''' : x \to \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2!}{(1 + \frac{x}{2})^2}$ $\qquad f'''(0) = (\frac{1}{2})^3 \cdot 2!$

$\vdots \qquad\qquad\qquad\qquad \vdots$

$f^{(n-1)}(x) \to (-\frac{1}{2})^{n-1} \frac{(n - 1)!}{(1 + \frac{x}{2})^n}$ $\qquad f^{(n-1)}(0) = (-\frac{1}{2})^{n-1} \cdot (n - 1)!$

$$P_{n-1}(x) = \log_e 2 + \frac{x}{2} - \frac{1}{2}(\frac{x}{2})^2 + \frac{1}{3}(\frac{x}{2})^3 + \ldots \frac{(-1)^{n-2}}{n - 1}(\frac{x}{2})^{n-1}$$

$$R_{n-1} = \frac{(-1)^{n-1}(\frac{x}{2})^n}{n(1 + \frac{x}{2})}$$

The limit of $R_{n-1}$ as $n \to \infty$ is dependent upon $(\frac{x}{2})^n$. If $|\frac{x}{2}| > 1$ then $\lim_{n \to \infty} R_{n-1} \to \infty$; if $|\frac{x}{2}| < 1$ then $\lim_{n \to \infty} R_{n-1} \to 0$. Thus, $0 < x < 2$ is necessary for $R_{n-1} \to 0$ as $n \to \infty$.

8. (a) $\tan(\alpha + \beta) = \dfrac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$

Let $\alpha = \arctan \frac{1}{2}$

and $\beta = \arctan \frac{1}{3}$

Then $\tan(\alpha + \beta) = \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = \dfrac{\frac{5}{6}}{\frac{5}{6}} = 1.$

Thus $\alpha + \beta = \frac{\pi}{4}$.

(b) $\pi = 4(\arctan \frac{1}{2} + \arctan \frac{1}{3})$

The remainder term (22) gives us our estimate of error.

$$|R_n| = |(-1)^n \int_0^x \frac{t^{2n}}{1 + t^2} dt| \le |\int_0^x t^{2n} dt|$$

$$|R_n| \le \frac{x^{2n+1}}{2n + 1}$$

Since two decimal place accuracy is required then

$$4(R_n(\arctan \tfrac{1}{2}) + R_n(\arctan \tfrac{1}{3})) \le 5 \times 10^{-3}$$

also $\qquad R_n(\arctan \frac{1}{2}) \le \dfrac{(\frac{1}{2})^{2n+1}}{2n + 1}$

and $\qquad R_n(\arctan \frac{1}{3}) \le \dfrac{(\frac{1}{3})^{2n+1}}{2n + 1}$

finally $\qquad \dfrac{4}{2n + 1} [(\frac{1}{2})^{2n+1} + (\frac{1}{3})^{2n+1}] \le 5 \times 10^{-3}$.

We simplify this inequality to obtain

$$8 \times 10^2(3^{2n+1} + 2^{2n+1}) \le (2n + 1)6^{2n+1}.$$

Try $n = 3$; $1,852,000 \le 1,959,552$. Since $n = 3$ just barely satisfies the inequality we will use $n = 4$ to insure success.

$$\pi = 4[\arctan\frac{1}{2} + \arctan\frac{1}{3}]$$

$$\approx 4[\frac{1}{2} - \frac{1}{24} + \frac{1}{160} - \frac{1}{896} + \frac{1}{3} - \frac{1}{81} + \frac{1}{1215} - \frac{1}{15309}]$$

$$\approx 3.14096 = 3.14 \text{ to two decimal places.}$$

9. (a) Let $\alpha = 4 \arctan\frac{1}{5}$ and $\beta = \arctan\frac{1}{239}$.

Let $\theta = \arctan\frac{1}{5}$

$$\tan\alpha = \tan(2\theta + 2\theta)$$

$$\tan 2\theta = \frac{2 \cdot \frac{1}{5}}{1 - \frac{1}{25}} = \frac{5}{12}$$

$$\tan\alpha = \tan 2(2\theta) = \frac{2 \cdot \frac{5}{12}}{1 - \frac{25}{144}} = \frac{120}{119}$$

$$\tan(\alpha - \beta) = \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} = \frac{\frac{28561}{28441}}{\frac{28561}{28441}} = 1$$

Thus $(\alpha - \beta) = \frac{\pi}{4}$.

(b) Before actually calculating $\pi$ we must investigate the error estimate.

Let $R$ be the error estimate for the Taylor approximation of $\arctan\frac{1}{5}$ and $S$ be the error estimae for $\arctan\frac{1}{239}$. The total error estimate must be less than $5 \times 10^{-3}$. Since $4 \cdot (\frac{4}{\pi}) = 4[4 \arctan\frac{1}{5} - \arctan\frac{1}{239}]$ the maximum error will be $4(4|R| + |S|)$. We do not have to use the same $n$ for both $R$ and $S$.

$$|R| \le \left|\frac{(\frac{1}{5})^{2n+1}}{2n + 1}\right| \quad\text{and}\quad |S| \le \left|\frac{(\frac{1}{239})^{2n+1}}{2n + 1}\right|$$

As a trial let $n = 1$ to calculate $S$.

$$|S|_{n=1} \le \frac{(\frac{1}{239})^3}{3} \approx 2 \cdot 4 \times 10^{-8}$$

Let us try $n = 3$ for $R$.

$$|R| \le \frac{(\frac{1}{5})^7}{7} \approx 1.828 \times 10^{-6}$$

$$4|R| \le 7.312 \times 10^{-6} \approx 731.2 \times 10^{-8}$$

$$4|R| + |S| \le 733.6 \times 10^{-8} \approx 8 \times 10^{-6}$$

The total error estimate can now be calculated.

$$4[4|R| + |S|] \le 3.2 \times 10^{-5} < 5 \times 10^{-2}.$$

$$\pi \approx 4[4(\tfrac{1}{5} - \tfrac{1}{3}(\tfrac{1}{5})^3 + \tfrac{1}{5}(\tfrac{1}{5})^5) - (\tfrac{1}{239})]$$

$$\approx 3.14152 \approx 3.14 \text{ to two decimal places.}$$

Obviously, we used more terms than were necessary.

10. (a) Show that $\log_e 2 = -7 \log_e \frac{9}{10} + 2 \log_e \frac{24}{20} + 3 \log_e \frac{81}{80}$.

This is true if $2 = (\frac{9}{10})^7 \cdot (\frac{24}{25})^2 \cdot (\frac{81}{80})^3$

$$= \frac{2^{13} \cdot 5^7 \cdot 3^{14}}{2^{12} \cdot 5^7 \cdot 3^{14}} = 2$$

(b) Each number $\frac{9}{10}$, $\frac{24}{25}$ and $\frac{81}{80}$ is very close to 1. The greatest error in any one of these Taylor approximations of the logarithms will occur for $x = -\frac{1}{10}$; call this error $R$.

The maximum possible accumulated error obtainable from summing the three series will be greater than $|R| + 2|R| + 3|R| = 12|R|$. If five decimal place accuracy is required then

$$12|R| \le 5 \times 10^{-6}$$

must be satisfied.

Let $\quad |R| \le \frac{(\frac{1}{10})^{n+1}}{n+1}$ from (23).

Then $\quad \frac{12(\frac{1}{10})^{n+1}}{n+1} \le 5 \times 10^{-6} < 6 \times 10^{-6}$

or $\quad \frac{2}{10^{n-5}} < n + 1.$

We find that $n \ge 5$ is the integer solution of this inequality.

11. Since $f : x \to \log \frac{1 + x}{1 - x}$ then also $f : x \to \log_e(1 + x) - \log_e(1 - x)$

$$p_n(x) \approx \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots (-1)^{n-1} \frac{x^n}{n} + (R_1)_n\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} \ldots \frac{x^n}{n} + (R_2)_n\right)$$

$$p_n(x) \approx 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots + \frac{x^{2n-1}}{2n - 1} + R_n\right)$$

$$R_n = (R_1)_n - (R_2)_n = (-1)^n \int_0^x \frac{t^n}{1 + t}\, dt - \int_0^x \frac{t^n}{1 - t}\, dt$$

$$R_n = \int_0^x \frac{t^n(-2t)}{1 - t^2}\, dt \quad \text{if } n \text{ is even.}$$

$$R_n = \int_0^x \frac{-2t^n}{1 - t^2}\, dt \quad \text{if } n \text{ is odd.}$$

Teacher's Commentary

Appendix 3

MATHEMATICAL INDUCTION

TC A3-1. The Principle of Mathematical Induction

The Principle of Mathematical Induction may be thought of as a postulate for the set of natural numbers $N$, rather than as a postulate about legitimate methods of proof (Metamathematics). Thus, we may state the principle in the following form:

Let $M$ be a subset of $N$ satisfying

(i) $1 \in M$,

(ii) if $n \in M$, then $n + 1 \in M$,

then $M = N$.

We can then deduce the form of the principle in the text by setting $M$ equal to the set of natural numbers $n$ for which $A_n$ is true.

As stated here, the Principle of Mathematical Induction can be used to play a central role in the axiomatic development of the natural numbers. In Foundations of Analysis by E. Landau (Chelsea), the arithmetic of the natural, rational, real, and complex numbers is developed solely on the basis of the five postulates of Peano. The fifth postulate is the postulate of induction.

Solutions Exercises A3-1

The solutions of several of the exercises follow the same pattern for the sequential step. In each case after assuming $A_k$, we add an appropriate term to each side of the equation which is the expression of $A_k$, and show that the resulting equation reduces to $A_{k+1}$. For brevity we give only the solutions of two such exercises; these will be found below in 2 and 12.

1. Prove by mathematical induction that $1 + 2 + 3 + \dots + n = \frac{1}{2}n(n + 1)$.

Follow the pattern given in 2.

2. By mathematical induction prove the familiar result, giving the sum of an arithmetic progression to $n$ terms:

$$a + (a + d) + (a + 2d) + \ldots + \left(a + (n - 1)d\right) = \frac{n}{2}[2a + (n - 1)d].$$

Initial Step. $a = \frac{1}{2}(2a + 0 \cdot d) = a$

Sequential Step. Assume $A_k$ : $a + (a + d) + (a + 2d) + \ldots + [a + (k - 1)d]$ $= \frac{k}{2}[2a + (k - 1)d]$.

Add $a + kd$ to both sides getting

$$\begin{aligned} a + (a + d) + \ldots + [a + (k - 1)d] + (a + kd) &= \frac{k}{2}[2a + (k - 1)d] + (a + kd) \\ &= ka + \frac{k(k - 1)}{2} d + a + kd \\ &= (k + 1)a + \frac{k^2 - k + 2k}{2} d \\ &= \frac{(k + 1)}{2}(2a + kd) \\ &= \frac{k + 1}{2} \{2a + [(k + 1) - 1]d\} \end{aligned}$$

which is $A_{k+1}$.

This completes the proof.

3. By mathematical induction prove the familiar result, giving the sum of a geometric progression to $n$ terms:

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}.$$

Follow the pattern given in 2.

Prove the following four statements by mathematical induction.

4. $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{1}{3}(4n^3 - n)$

Follow the pattern given in 2.

5. $2n \leq 2^n$

<u>Initial</u> <u>Step</u>. $2 \cdot 1 \leq 2^1$

<u>Sequential</u> <u>Step</u>. Assume $A_k: 2 \cdot k \leq 2^k$.

Then $\qquad 2(k + 1) = 2k + 2 \leq 2k + 2k, \qquad$ since $k \geq 1$.

$$\leq 2 \cdot 2k \leq 2 \cdot 2^k,$$

by the assumption $A_k$.

Therefore $2(k + 1) \leq 2^{k+1}$ which is $A_{k+1}$.

This completes the proof.

6. If $p > -1$; then, for every positive integer $n$, $(1 + p)^n \geq 1 + np$.

<u>Initial</u> <u>Step</u>. $(1 + p)^1 \geq 1 + 1 \cdot p$

<u>Sequential</u> <u>Step</u>. Assume $A_k : (1 + p)^k \geq 1 + kp$.

Then, since $p > -1$, $1 + p > 0$, and we may multiply both sides of the inequality by $1 + p$ without changing its sense.

Therefore $(1 + p)^{k+1} \geq (1 + kp)(1 + p)$

$\geq 1 + kp + p + kp^2$ and dropping the positive quantity $kp^2$ we get $(1 + p)^{k+1} \geq 1 + (k + 1)p$ which is $A_{k+1}$.

This completes the proof.

7. $1 + 2 \cdot 2 + 3 \cdot 2^2 + \ldots + n \cdot 2^{n-1} = 1 + (n - 1)2^n$

Follow the pattern given in 2.

Prove the following by the second principle of mathematical induction.

8. For all natural numbers $n$, the number $n + 1$ either is a prime or can be factored into primes.

We use the second principle of induction.

<u>Initial</u> <u>Step</u>. The number 2 is a prime.

Sequential Step. Let $A_n$ be the statement of Exercise 8, and assume that $A_s$ is true for all natural numbers $s$ satisfying $s \leq k$. In other words, every integer 2, 3, 4, ..., $k + 1$ is prime or a product of primes. In order to prove $A_{k+1}$ we must show that $k + 2$ either is prime or can be factored into primes. If $k + 2$ is prime we are done. If not, we can write $k + 2$ as a product of factors $r$ and $t$ both less than $k + 2$, hence, both less than or equal to $k + 1$. By hypothesis, then, both factors $r, t$ must be primes or products of primes. It follows that $k + 2$ can be written as a product of primes and that $A_{k+1}$ is true.

9. For each natural number $n$ greater than one, let $U_n$ be a real number with the property that for at least one pair of natural numbers $p, q$ with $p + q = n$, $U_n = U_p + U_q$.

When $n = 1$, we define $U_1 = a$ where $a$ is some given real number. Prove that $U_n = na$ for all $n$.

Initial Step. $U_1 = 1 \cdot a$ by definition.

Sequential Step. Let $A_n$ be the statement of Exercise 9. Using the second principle of induction we assume that for each number $s \leq k$ that $A_s$ is true.

Now $A_{k+1}$ must be established. But if $U_{k+1}$ is a real number such that for at least one pair of natural numbers, $f$ and $g$ such that $f + g = k + 1$,

$$U_{k+1} = U_f + U_g,$$

we know that $f$ and $g$ must each be less than or equal to $k$; and therefore $U_f$ and $U_g$ are real numbers to which the sequential hypothesis may be applied. Therefore

$$U_f = f \cdot a \text{ and } U_g = g \cdot a$$

and so

$$U_{k+1} = f \cdot a + g \cdot a = (f + g) \cdot a = (k + 1)a$$

which is $A_{k+1}$.

This completes the proof.

10. Attempt to prove 8 and 9 from the first principle to see what difficulties arise.

In 8 note that the sequential step is based essentially upon the fact that $r$ and $t$ are each at most $k + 1$, not necessarily equal to $k + 1$. It would therefore be impossible to derive $A_{k+1}$ from $A_k$ alone and we cannot employ the first principle. Similarly in 9, we know only that $f$ and $g$ are at most $k$, not that they are necessarily equal to $k$. So we need to be able to refer to $A_s$ for $s \le k$, not to just $A_k$.

In the next three problems, first discover a formula for the sum, and then prove by mathematical induction that you are correct.

11. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n + 1)}$

To discover the formula for the sum, we might try writing down the sums in succession.

Thus

$$S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

$$S_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$S_3 = \frac{2}{3} + \frac{1}{3 \cdot 4} = \frac{3}{4}$$

$$S_4 = \frac{3}{4} + \frac{1}{4 \cdot 5} = \frac{4}{5}$$

So we guess $S_n = \frac{n}{n + 1}$ and try to prove it.

For the proof follow the pattern of 2.

12. $1^3 + 2^3 + 3^3 + \dots + n^3$. (Hint: Compare the sums you get here with Examples A3-1a and A3-1g in the text, or, alternatively, assume that the required result is a polynomial of degree 4.)

To guess the sum, we write down in succession the following:

$$S_1 = 1^3 = 1 = 1^2 = 1^2;$$

$$S_2 = 1 + 2^3 = 9 = 3^2 = (1 + 2)^2;$$

$$S_3 = 9 + 3^3 = 36 = 6^2 = (1 + 2 + 3)^2;$$

$$S_4 = 36 + 4^3 = 100 = 10^2 = (1 + 2 + 3 + 4)^2.$$

We guess therefore that $S_n = (1 + 2 + 3 + \ldots + n)^2$. To prove this directly by induction is quite messy (try it), but if we remember from Number 1 that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$, we get a formula much easier to prove: $1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$.

Now we follow the pattern of 2.

Initial Step. $1^3 = \frac{1^2 \cdot (1+1)^2}{4} = 1$

Sequential Step. Assume $A_k : 1^3 + 2^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}$.

Add $(k+1)^3$ to both sides, getting the following:

$$\begin{aligned} 1^3 + 2^3 + \ldots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{[(k+1)^2(k^2 + 4(k+1))]}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \frac{(k+1)^2((k+1)+1)^2}{4}, \end{aligned}$$

which is $A_{k+1}$.

This completes the proof.

13. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1)$. (Hint: Compare this with Example A3-1g in the text.)

To guess the sum we write down in succession the following:

$$\begin{aligned} S_1 &= 1 \cdot 2 &&= 2 \\ S_2 &= 2 + 2 \cdot 3 &&= 8 \\ S_3 &= 8 + 3 \cdot 4 &&= 20 \\ S_4 &= 20 + 4 \cdot 5 &&= 40. \end{aligned}$$

This does not seem to be getting us very far. We try another approach. If you have worked Example A3-1g (and remember it) try writing $S_n$ in this fashion,

$$\begin{aligned}
S_n &= 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) \\
&= 1(1+1) + 2(2+1) + 3(3+1) + \ldots + n(n+1) \\
&= 1^2 + 1 + 2^2 + 2 + 3^2 + 3 + \ldots + n^2 + n \\
&= (1^2 + 2^2 + 3^2 + \ldots + n^2) + (1 + 2 + 3 + \ldots + n) \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1+3)}{6} \\
&= \frac{n(n+1)(n+2)}{3} .
\end{aligned}$$

Another way of guessing this formula would be to assume, as in Example A3-1g, that since the general term in $S_n$ is quadratic, the formula might be cubic

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = an^3 + bn^2 + cn + d$$

and then let $n$ take on the successive values 1, 2, 3, and 4 to determine $a$, $b$, $c$, and $d$. Thus, by successive subtractions,

$$\left.\begin{aligned}
a + b + c + d &= 2 \\
8a + 4b + 2c + d &= 8 \\
27a + 9b + 3c + d &= 20 \\
64a + 16b + 4c + d &= 40
\end{aligned}\right\}
\quad
\left.\begin{aligned}
7a + 3b + c &= 6 \\
19a + 5b + c &= 12 \\
37a + 7b + c &= 20
\end{aligned}\right\}
\quad
\left.\begin{aligned}
12a + 2b &= 6 \\
18a + 2b &= 8
\end{aligned}\right\}
\quad 6a = 2.$$

Therefore,

$$a = \frac{1}{3}, \quad b = 1, \quad c = \frac{2}{3}, \quad d = 0,$$

and

$$S_n = \frac{1}{3}n^3 + n^2 + \frac{2}{3}n = \frac{n(n+1)(n+2)}{3} .$$

The proof of these results follows the pattern of 2 and 12.

14. Prove for all positive integers $n$,

$$\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right) \cdots \left(1 + \frac{2n+1}{n^2}\right) = (n+1)^2 .$$

Check the initial step.

Assume $A_k$, and multiply both sides of the resulting equation by the appropriate factor, and reduce to get $A_{k+1}$.

15. Prove that $(1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2^n}) = \frac{1 - x^{2^{n+1}}}{1 - x}$.

Follow the pattern of Solution 14.

16. Prove that $n(n^2 + 5)$ is divisible by 6 for all integral $n$.

Initial Step. $1(1 + 5) = 6$ and this is divisible by 6.
Assume $A_k : k(k^2 + 5) = 6p$ where $p$ is a positive integer.
Consider:

$$\begin{aligned}(k + 1)\left((k + 1)^2 + 5\right) &= (k + 1)^3 + 5(k + 1) \\ &= k^3 + 3k^2 + 3k + 1 + 5k + 5 \\ &= (k^3 + 5k) + (3k^2 + 3k) + 1 + 5 \\ &= k(k^2 + 5) + 3k(k + 1) + 6.\end{aligned}$$

By $A_k$ we know that $k(k^2 + 5) = 6p$, and since $k$ is a positive integer either $k$ or $k + 1$ is an even integer. Therefore the second term is divisible both by 2 and by 3, and therefore by 6. Finally we get

$$\begin{aligned}(k + 1)\left((k + 1)^2 + 5\right) &= 6p + 6q + 6 \\ &= 6(p + q + 1)\end{aligned}$$

and this finishes the proof, since we know that the sum of three positive integers is a positive integer.

17. Any infinite straight line separates the plane into two parts; two intersecting straight lines separate the plane into four parts; and three non-concurrent lines, of which no two are parallel, separate the plane into seven parts. Determine the number of parts into which the plane is separated by $n$ straight lines of which no three meet in a single common point and no two are parallel; then prove your result. Can you obtain a more general result when parallelism is permitted? If concurrence is permitted? If both are permitted?

Both our method of guessing the answer and our proof will be sequential. Let $R_n$ be the number of regions into which the plane is divided by $n$ lines of which no two are parallel and no three are concurrent. If we draw an $(n + 1)$-th line under the same conditions, it must meet all the other lines in $n$ new points of intersection. In crossing $n$ lines it must go through $n + 1$ regions of the plane, dividing each region into two parts, thus adding $n + 1$ new regions. We conclude that

$$R_{n+1} = (n + 1) + R_n.$$

Since $R_1 = 2$, this is a recursive definition for $R_n$. We have, plainly,

$$R_n = 2 + 2 + 3 + 4 + \dots + n = \frac{1}{2}(n^2 + n + 2)$$

and this result can be obtained directly from the recursion formula by a straightforward induction.

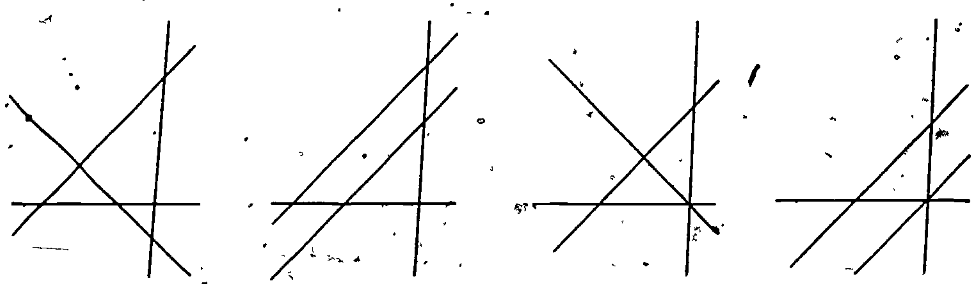
If parallelism is permitted, each pair of parallel lines existing reduces $R_n$ by 1, since one crossing is eliminated. Thus if $p$ lines are parallel, you can pick $\frac{p(p - 1)}{2}$ pairs of parallel lines and there will be this many fewer regions

$$R_n = 1 + \frac{n(n + 1)}{2} - \frac{(p - 1)p}{2}.$$

For example if four lines are drawn, three of which are parallel, there will be

$$1 + \frac{4(5)}{2} - \frac{3(2)}{2} = 8 \text{ regions}.$$

Similarly, any line which concurs with an already existing intersection point reduces the total number of intersection points by one, and the number of regions of the plane by one. Again we must remember, as in



n=4, p=0, c=0 $R_4$=11 | n=4, p=2, c=0 $R_4$=10 | n=4, p=0, c=3 $R_4$=10 | n=4, p=2, c=3 $R_4$=9

the parallel case, that pairs of extra concurrencies must all be counted. Thus if $c$ lines concur at one point

$$R_n = 1 + \frac{n(n + 1)}{2} - \frac{(c - 1)(c - 2)}{2}.$$

If a line provides both a case of parallelism and a case of concurrence, it must be counted each way in reducing the number of regions, as is shown in the figure. In general if there are $j$ families of parallel lines with $p_1, p_2, \ldots, p_j$ lines in each family and $k$ families of concurrent lines with $c_1, c_2, \ldots, c_k$ lines in each family, we have

$$R_n = 1 + \frac{n(n+1)}{2}$$
$$- \left[\frac{p_1(p_1 - 1)}{2} + \frac{p_2(p_2 - 1)}{2} + \ldots + \frac{p_j(p_j - 1)}{2}\right]$$
$$- \left[\frac{(c_1 - 1)(c_1 - 2)}{2} + \frac{(c_2 - 1)(c_2 - 2)}{2} + \ldots + \frac{(c_k - 1)(c_k - 2)}{2}\right]$$

The proof of this is too lengthy for insertion here.

18. Consider the sequence of fractions

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{p_n}{q_n}, \ldots$$

where each fraction is obtained from the preceding by the rule

$$p_n = p_{n-1} + 2q_{n-1}$$
$$q_n = p_{n-1} + q_{n-1}.$$

Show that for $n$ sufficiently large, the difference between $\frac{p_n}{q_n}$ and $\sqrt{2}$ can be made as small as desired. Show also that the approximation to $\sqrt{2}$ is improved at each successive stage of the sequence and that the error alternates in sign. Prove also that $p_n$ and $q_n$ are relatively prime, that is, the fraction $\frac{p_n}{q_n}$ is in lowest terms.

Let the error at the n-th stage be denoted by $e_n = \frac{p_n}{q_n} - \sqrt{2}$. We may define the error $e_{n+1}$ at the next stage recursively in terms of $e_n$ as follows:

$$e_{n+1} = \frac{p_n + 2q_n}{p_n + q_n} - \sqrt{2}$$

$$= \frac{\frac{p_n}{q_n} + 2}{\frac{p_n}{q_n} + 1} - \sqrt{2}$$

$$= \frac{e_n + 2 + \sqrt{2}}{e_n + 1 + \sqrt{2}} - \sqrt{2}$$

$$= \frac{e_n(1 - \sqrt{2})}{e_n + 1 + \sqrt{2}}.$$

Since $1 - \sqrt{2}$ is negative, it follows that $e_{n+1}$ has the opposite sign from $e_n$, and the sign alternates if the denominator is shown to be positive. We shall prove by induction that $|e_n| < \frac{1}{2^n}$ and thereby show simultaneously that the denominator above is positive, and that the error can be made as small as desired by taking $n$ sufficiently large.

<u>Initial Step</u>. $|e_1| = |1 - \sqrt{2}| = .414\ldots < \frac{1}{2}$

<u>Sequential Step</u>. Assume $|e_k| < \frac{1}{2^k}$. For the denominator of $e_{k+1}$, we have

$$e_k + 1 + \sqrt{2} > -\frac{1}{2^k} + 1 + \sqrt{2} > -\frac{1}{2} + 1 + \sqrt{2} > \frac{1}{2} + \sqrt{2} > 1.$$

We also have
$$\sqrt{2} - 1 < \frac{1}{2}.$$

It follows from the recursive expression for $e_{k+1}$ that

$$|e_{k+1}| < \frac{1}{2}|e_k| < \frac{1}{2} \cdot \frac{1}{2^k} = \frac{1}{2^{k+1}}.$$

To prove that $p_n$ and $q_n$ have no common factor other than 1, we note that

$$p_n = p_{n+1} - q_{n+1}, \quad q_n = 2q_{n+1} - p_{n+1}.$$

We then reason inductively as follows:

<u>Initial Step</u>. The only common factor of $p_1$ and of $q_1$ is 1.

<u>Sequential Step</u>. Assume $p_k$ and $q_k$ have no common factor other than 1. If $p_{k+1}$ and $q_{k+1}$ had such a common factor, then, by the above formula it would have to be a common factor of $p_k$ and $q_k$. Contradiction.

19. Let $p$ be any polynomial of degree $m$. Let $q(n)$ denote the sum

(1) $$q(n) = p(1) + p(2) + p(3) + \ldots + p(n).$$

Prove that there is a polynomial $q$ of degree $m + 1$ satisfying (1).

Initial Step. We observe that if $p$ has degree 0, then $p = c$ where $c$ is a constant and we have

(1) $$p(1) + p(2) + \ldots + p(n) = c + c + c + \ldots + c = cn.$$

Hence $q(x) = cx$ is a polynomial of first degree satisfying the condition.

Sequential Step. We assume that the theorem is true for any polynomial $p$ of degree less than or equal to $k$. Let

(2) $$p(x) = ax^{k+1} + p_1(x), \qquad (a \neq 0)$$

where the degree of $p_1$ is $\leq k$.

Next we observe that

(3) $$(x + 1)^{k+2} = x^{k+2} + (k + 2)x^{k+1} + p_2(x)$$

where the degree of $p_2$ is $\leq k$. This fact has to be proved by induction, unless the binomial theorem is taken for granted. It will be proved afterward. Setting

$$(x + 1)^{k+2} - x^{k+2} = (k + 2)x^{k+1} + p_2(x)$$

and solving for $x^{k+1}$ we obtain in (2)

(4) $$p(x) = \frac{a}{k + 2}[(x + 1)^{k+2} - x^{k+2}] + p_3(x)$$

where $$p_3(x) = p_1(x) - \frac{a}{k + 2}p_2(x)$$

and therefore the degree of $p_3(x)$ is $\leq k$.

Consequently,

$$p(1) + p(2) + \ldots + p(n) = \frac{a}{k + 2}\{(2^{k+2} - 1^{k+2}) + (3^{k+2} - 2^{k+2}) + \ldots + [(n + 1)^{k+2} - n^{k+2}]\} + p_3(1) + p_3(2) + \ldots + p_3(n).$$

By the induction hypothesis, there exists a $q_1(x)$ of degree $\leq k + 1$, such that

$$q_1(n) = p_3(1) + \ldots + p_3(n).$$

Furthermore the expression in braces reduces by successive additions and subtractions to $(n+1)^{k+2} - 1^{k+2}$, and we obtain the desired polynomial,

$$q(x) = \frac{a}{k+2}[(x+1)^{k+2} - 1^{k+2}] + q_1(x)$$

where $q(n) = p(1) + \dots + p(n)$.

Now we prove (3):

Initial Step. If $k = 0$, $(x+1)^2 = x^2 + (0+2)x + 1$ and the degree of 1 is 0.

Sequential Step. $(x+1)^{k+3} = x^{k+2}(x+1) + (k+2)x^{k+1}(x+1) + (x+1)p_2(x)$

$$= x^{k+3} + (k+3)x^{k+2} + [(k+2)x^{k+1} + (x+1)p_2(x)]$$

20. Let the function $f(n)$ be defined recursively as follows:
Initial Step. $f(1) = 3$
Sequential Step. $f(n+1) = 3^{f(n)}$
In particular, we have $f(3) = 3^{3^3} = 3^{27}$, etc.
Similarly, $g(n)$ is defined by
Initial Step. $g(1) = 9$
Sequential Step. $g(n+1) = 9^{g(n)}$
Find the minimum value $m$ for each $n$ such that $f(m) \geq g(n)$.

It is easily seen that $g(n) > f(n)$ for all $n$. We shall prove that $f(n+1) > g(n)$ for all $n$ and, hence, that $m = n+1$ is the least value for which $f(m) > g(n)$.

Initial Step. If $n = 1$, $f(2) = 3^3 = 27$, and $g(1) = 9$. Consequently $f(2) > g(1)$. More strongly, $f(2) > 2g(1) + 1$; and we shall prove generally $f(n+1) > 2g(n) + 1$.

Sequential Step. Suppose:

$$f(k+1) > 2g(k) + 1 > g(k).$$

Then,

$$\begin{aligned} f(k+2) = 3^{f(k+1)} &> 3^{2g(k)+1} \geq 3 \cdot 3^{2g(k)} \\ &\geq 3 \cdot 9^{g(k)} \geq 3g(k+1) \\ &> 2g(k+1) + 1 \\ &> g(k+1). \end{aligned}$$

21. Prove for all natural numbers $n$ that $\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$ is an integer. (Hint: Try to express $x^n - y^n$ in terms of $x^{n-1} - y^{n-1}$, $x^{n-2} - y^{n-2}$, etc.)

Let $F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$. We will use the second principle.

Initial Step. $F_1 = \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} = 1$

$$F_2 = \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} = \frac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{4\sqrt{5}}$$

$$= \frac{4\sqrt{5}}{4\sqrt{5}} = 1.$$

Sequential Step. Assume $F_s$ is an integer for all $s \leq k$.

Consider $F_{k+1} = \frac{(1+\sqrt{5})^{k+1} - (1-\sqrt{5})^{k+1}}{2^{k+1}\sqrt{5}}$. For brevity we write

$$1 + \sqrt{5} = x \quad \text{and} \quad 1 - \sqrt{5} = y.$$

Then

$$F_{k+1} = \frac{x^{k+1} - y^{k+1}}{2^{k+1}\sqrt{5}} = \frac{x^{k+1} + x^k y - x^k y + xy^k - xy^k - y^{k+1}}{2^{k+1}\sqrt{5}}$$

$$= \frac{x^k(x+y) - xy(x^{k-1} - y^{k-1}) - y^k(x+y)}{2^{k+1}\sqrt{5}}$$

$$= \frac{(x+y)(x^k - y^k)}{2^{k+1}\sqrt{5}} - \frac{xy(x^{k-1} - y^{k-1})}{2^{k+1}\sqrt{5}}$$

$$= \frac{x+y}{2}\left(\frac{x^k - y^k}{2^k\sqrt{5}}\right) - \frac{xy}{4}\left(\frac{x^{k-1} - y^{k-1}}{2^{k-1}\sqrt{5}}\right)$$

$$= \frac{(1+\sqrt{5}) + (1-\sqrt{5})}{2}F_k - \frac{(1+\sqrt{5})(1-\sqrt{5})}{4}F_{k-1}$$

$$= F_k + F_{k-1}.$$

but by the assumption of the sequential step we know $F_k$ and $F_{k-1}$ are integers. Therefore $F_{k+1}$ is an integer. This completes the theorem.

## Solutions Exercises A3-2a

1. Prove

$$\sum_{k=1}^{n} (\alpha f_k + \beta g_k) = \alpha \sum_{k=1}^{n} f_k + \beta \sum_{k=1}^{n} g_k$$

The linearity of summation is a consequence of the additive and multiplicative properties of real numbers and follows easily by mathematical induction.

2. Write each of the following sums in expanded form and evaluate:

(a) $\sum_{k=1}^{5} 2k$ $\quad 2 + 4 + 6 + 8 + 10 = 30$

(b) $\sum_{j=2}^{6} j^2$ $\quad 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 90$

(c) $\sum_{r=-1}^{3} (r^2 + r - 12)$ $\quad (-12) + (-12) + (-10) + (-6) + (0) = -40$

(d) $\sum_{m=2}^{5} m(m-1)(m-2)$ $\quad 0 + 3 \cdot 2 \cdot 1 + 4 \cdot 3 \cdot 2 + 5 \cdot 4 \cdot 3 = 90$

(e) $\sum_{i=0}^{10} 2^i$ $\quad 1 + 2 + 2^2 + 2^3 + \ldots + 2^{10} = 2^{11} - 1 = 2043$

(f) $\sum_{r=0}^{4} \frac{4!}{r!(4-r)!}$ $\quad 1 + 4 + 6 + 4 + 1 = 16$

3. Which of the following statements are true and which are false? Justify your conclusions.

(a) $\displaystyle\sum_{j=3}^{10} 4 = 7 \cdot 4 = 28$

(b) $\displaystyle\sum_{j=m}^{n} 4 = 4(n - m + 1)$

(c) $\displaystyle\sum_{k=1}^{10} k^2 = 10\sum_{k=1}^{9} k^2$

(d) $\displaystyle\sum_{k=1}^{1000} k^2 = 5 + \sum_{k=3}^{1000} k^2$

(e) $\displaystyle\sum_{k=1}^{n} k^3 = n^3 + \sum_{j=2}^{n} (j - 1)^3$

(f) $\displaystyle\sum_{m=1}^{10} k^2 = \left(\sum_{m=1}^{10} k\right)^2$

(g) $\displaystyle\sum_{m=1}^{10} k^3 = \left(\sum_{m=1}^{10} k\right)^2$

(h) $\displaystyle\sum_{i=0}^{n} i(i - 1)(n - i) = \sum_{i=2}^{n-1} i(i - 1)(n - i)$

(i) $\displaystyle\sum_{k=0}^{m} f(a_{m-k}) = \sum_{k=0}^{m} f(a_k)$

(j) $\displaystyle n\sum_{k=0}^{n} A_k - \sum_{k=0}^{n} kA_k = \sum_{k=0}^{n} kA_{n-k}$

(k) $\displaystyle\sum_{k=0}^{m} k^2(A_k - A_{m-k}) = m^2\sum_{k=0}^{m} A_{m-k} - 2m\sum_{k=0}^{m} k\, A_{m-k}$

(a) False; $\sum_{j=3}^{10} 4 = 8 \cdot 4 = 32$

(b) True

(c) False; $\sum_{k=1}^{10} k^2 = 10^2 + \sum_{k=1}^{9} k^2 < 10 \sum_{k=1}^{9} k^2$

(d) True

(e) True

(f) False; unless $k = 0$.

(g) False; unless $k = 0$.

(h) True; the missing terms are zero.

(i) True; $m - k$ takes on the same values as $k$.

(j) True: $\sum_{k=0}^{n} k\,A_{n-k} = \sum_{k=0}^{n} (n - k)A_k$ by (i).

(k) True; follows by applying (i) to $\sum_{k=0}^{m} (m^2 - 2mk + k^2)A_{m-k}$.

4. Evaluate $\sum_{k=1}^{n} f(\frac{k}{n})(\frac{b - a}{n})$ if $f(x) = x^2$, $a = 0$, $b = 1$ and

(a) $n = 2$ $\quad \frac{5}{8}$

(b) $n = 4$ $\quad \frac{15}{32}$

(c) $n = 8$ $\quad \frac{102}{256}$

5. Subdivide the interval $[0,1]$ into $n$ equal parts. In each subinterval obtain upper and lower bounds for $x^2$. Using sigma notation, use these upper and lower bounds to obtain expressions for upper and lower estimates of the area under the curve $y = x^2$ on $[0,1]$. If you can evaluate these sums without reading elsewhere, do so.

$$\text{Lower sum} = \sum_{k=0}^{n-1} \frac{1}{n}\left(\frac{k}{n}\right)^2 = \frac{1}{n^3}\sum_{k=0}^{n-1} k^2 = \frac{(n-1)(n)(2n-1)}{6n^3}.$$

$$\text{Upper sum} = \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k}{n}\right)^2 = \text{lower sum} + \frac{1}{n}.$$

6. (a) Write out the sum of the first 7 terms of an arithmetic progression with first term $a$ and common difference $d$. Express the same sum in sigma notation.

$$(a) + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + (a + 5d) + (a + 6d)$$

$$= \sum_{n=1}^{7} \{a + (n - 1)d\}.$$

(b) In sigma notation, write the expression for the sum of the first $n$ terms of a geometric progression with first term $a$ and common ratio $r$.

$$\sum_{k=1}^{n} ar^{k-1}.$$

7. (a) Consider a function $f$ defined by

$$f(n) = \sum_{r=1}^{n} \{(r - 1)(r - 2)(r - 3)(r - 4)(r - 5) + r\}.$$

Find $f(n)$ for $n = 1, 2, \ldots, 5$.

$$f(n) = \frac{n(n + 1)}{2}, \text{ for } n = 1, 2, \ldots, 5.$$

(b) Give an example of a function $g$ (similar to that in (a)) such that

$$g(n) = 1, \quad n = 1, 2, \ldots, 10^6,$$
$$g(10^6 + 1) = 0.$$

$$g(n) = 1 - \frac{(n-1)(n-2)(n-3)\ldots(n-10^6)}{10^6!}$$

8. Write each of the following sums in expanded form and evaluate:

(a) $$\sum_{n=1}^{4}\left\{\sum_{r=1}^{3} r(n-r)\right\}$$

$$\sum_{n=1}^{4}\{1(n-1) + 2(n-2) + 3(n-3)\} = \sum_{n=1}^{4}\{6n - 14\} = 4$$

(b) $$\sum_{n=1}^{N}\left\{\sum_{r=1}^{R}(rn - 1)\right\}$$

$$\sum_{n=1}^{N}\{(n-1) + (2n-1) + \ldots + (Rn-1)\} = \sum_{n=1}^{N}\frac{n(R)(R+1)}{2} - R$$

$$= \frac{N(N+1)(R)(R+1)}{4} - RN$$

9. The double sum $\sum_{i=0}^{m}\sum_{j=0}^{n} F(i,j)$ is a shorthand notation for

$$\sum_{i=0}^{m}\{F(i,0) + F(i,1) + \ldots + F(i,n)\} \quad \text{or} \quad \begin{aligned} &F(0,0) + F(0,1) + \ldots + F(0,n) \\ &+ F(1,0) + F(1,1) + \ldots + F(1,n) \\ &\quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ &+ F(m,0) + F(m,1) + \ldots + F(m,n). \end{aligned}$$

In particular,

$$\sum_{i=1}^{2}\sum_{j=1}^{3} i \cdot j = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 = 18$$

Evaluate:

(a) $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} i \cdot j$

(b) $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} (i + j)$

(c) $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} \max\{i,j\}$

(d) $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} \min\{i,j\}$

(a) $\displaystyle\sum_{i=1}^{m} i\left\{\sum_{j=1}^{n} j\right\} = \frac{m(m + 1)(n)(n + 1)}{4}$

(b) $\displaystyle\sum_{i=1}^{m} \{ni + \frac{n(n + 1)}{2}\} = \frac{n(m)(m + 1)}{2} + \frac{mn(n + 1)}{2} = \frac{mn(m + n + 2)}{2}$.

(c) If $n \geq m$, we have

$$\sum_{i=1}^{m}\left\{\sum_{j=1}^{i} i + \sum_{j=i+1}^{n} j\right\} = \sum_{i=1}^{m}\{i^2 + \frac{n(n + 1)}{2} - \frac{i(i + 1)}{2}\}$$

$$= \frac{1}{2}\sum_{i=1}^{m} i(i - 1) + \frac{1}{2}\sum_{i=1}^{m} n(n + 1)$$

$$= \frac{(m - 1)m(m + 1)}{6} + \frac{mn(n + 1)}{2}$$

For $m \geq n$, just interchange $m$ and $n$ (by symmetry).

(d) This can also be done similarly, i.e.,

$$\sum_{i=1}^{m}\left\{\sum_{j=1}^{i} j + \sum_{j=i+1}^{n} i\right\}$$

Alternatively,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \{\max(i,j) + \min(i,j)\} = \sum_{i=1}^{m}\sum_{j=1}^{n} (i + j)$$

Now use (b) and (c), to give

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \min(i,j) = \frac{m^2 n}{2} - \frac{(m-1)(m)(m+1)}{6}.$$

10. (a) Show that $\frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$, $k \neq 0, 1$.

$$\frac{1}{k-1} - \frac{1}{k} = \frac{k-(k-1)}{(k-1)k} = \frac{1}{k(k-1)}, \ k \neq 0.$$

(b) Evaluate $\sum_{k=2}^{1000} \frac{1}{k(k-1)}$.

$$\sum_{k=2}^{1000} \frac{1}{k(k-1)} = \sum_{k=2}^{1000} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{2-1} - \frac{1}{1000} = \frac{999}{1000}.$$

In general, $\sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 - \frac{1}{n}$.

11. If $S(n) = \sum_{i=1}^{n} f(i)$ determine $f(m)$ in terms of the sum function $S$.

$f(1) = S(1)$

$$S(m) - S(m-1) = \sum_{i=1}^{m} f(i) - \sum_{i=1}^{m-1} f(i) = f(m), \ m > 1.$$

12. Determine $f(m)$ in the following summation formulae: (See Number 11.)

(a) $1 = \sum_{i=1}^{n} f(i)$ $\qquad f(1) = 1, \ f(m) = 0, \ m > 1.$

(b) $n = \sum_{i=1}^{n} f(i)$  $f(m) = 1,\ m \geq 1.$

(c) $n^2 = \sum_{i=1}^{n} f(i)$  $f(m) = m^2 - (m-1)^2 = 2m - 1,\ m \geq 1.$

(d) $an^2 + bn + c = \sum_{i=1}^{n} f(i)$  $f(1) = a + b + c,$

$$f(m) = am^2 + bm + c - a(m-1)^2 - b(m-1) - c$$
$$= a(2m-1) + b,\ m > 1.$$

(e) $\cos nx = \sum_{i=1}^{n} f(i)$  $f(1) = \cos x,$

$$f(m) = \cos mx - \cos(m-1)x$$
$$= -2 \sin \frac{x}{2} \sin(m - \frac{1}{2})x,\ m > 1.$$

(f) $\sin(an + b) = \sum_{i=1}^{n} f(i)$  $f(1) = \sin(a + b),$

$$f(m) = \sin(am + b) - \sin\left(a(m-1) + b\right)$$
$$= 2 \sin \frac{a}{2} \cos\left(am + b - \frac{a}{2}\right),\ m > 1.$$

(g) $n! = \sum_{i=1}^{n} f(i)$  $f(1) = 1,$

$$f(m) = m! - (m-1)!$$
$$= (m-1)!(m-1),\ m > 1.$$

13. Binomial Theorem:

We define $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ where $r$, $n$ are integers such that $0 \leq r \leq n$. Also $0! = 1$ and $\binom{n}{r} = 0$ if $r > n$. Show that

(a) $\binom{n}{0} = \binom{n}{n} = 1$ $\qquad$ $\binom{n}{0} = \frac{n!}{n!0!} = \frac{n!}{n! \cdot 1} = 1$

$\binom{n}{1} = \binom{n}{n-1} = n$ $\qquad$ $\binom{n}{n} = \frac{n!}{0!n!} = 1$

$$\binom{n}{1} = \frac{n!}{(n-1)!1!} = \frac{n!}{(n-1)!(n-(n-1))!}$$

$$= \frac{n!}{(n-1)!} = n$$

(b) $\binom{n}{r} = \binom{n}{n-r}$ $\qquad$ $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-(n-r))!}$

$$= \binom{n}{n-r}$$

$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ $\qquad$ $\binom{n}{r} + \binom{n}{r+1}$

$$= \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r-1)!(r+1)!}$$

$$= \frac{n!(r+1)}{(n-r)!(r+1)!} + \frac{n!(n-r)}{(n-r)!(r+1)!}$$

$$= \frac{n!(n+1)}{(n-r)!(r+1)!} = \frac{(n+1)!}{(n-r)!(r+1)!}$$

$$= \binom{n+1}{r+1}$$

(c) Establish the Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r = x^n + nx^{n-1}y + \ldots + nxy^{n-1} + y^n, \; n = 0, 1, 2, \ldots$$

by mathematical induction.

For $n = 1$ ; $(x + y) = \binom{1}{0}xy^0 + \binom{1}{1}x^0y = x + y$.

Now assume $(x + y)^k = \sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^r$ for all $k \leq n$.

We show this implies the truth of the theorem for $k = n + 1$.

$$(x+y)^{n+1} = (x+y)^n(x+y) = \left(\sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r\right)(x+y)$$

$$= \sum_{r=0}^{n} \left(\binom{n}{r} x^{n-r+1} y^r + \binom{n}{r} x^{n-r} y^{r+1}\right)$$

$$= \sum_{r=1}^{n} \left(\left(\binom{n}{r} + \binom{n}{r-1}\right) x^{n-r+1} y^r\right) + \binom{n}{0} x^{n+1} y^0 + \binom{n}{n} x^0 y^{n+1}$$

(using (b))

$$= \sum_{r=1}^{n} \binom{n+1}{r} x^{n-r+1} y^r + \binom{n+1}{0} x^{n+1} y^0 + \binom{n+1}{n+1} x^0 y^{n+1}$$

$$= \sum_{r=0}^{n+1} \binom{n+1}{r} x^{n+1-r} y^r$$

$$= \sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^k, \quad \text{where } k = n+1.$$

14. Using the binomial theorem, give the expansions for the following:

(a) $(x+y)^3$

$$(x+y)^3 = \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

(b) $(x-y)^3$

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - 3y^3$$

(c) $(2x-3y)^3$

$$(2x-3y)^3 = \binom{3}{3}(2x)^3 + \binom{3}{2}(2x)^2(-3y) + \binom{3}{1}(2x)(-3y)^2 + \binom{3}{0}(-3y)^3$$
$$= 8x^3 - 36x^2y + 54xy^2 - 23y^3$$

(d) $(x-2y)^5$

$$(x-2y)^5 = \binom{5}{5}x^5 + \binom{5}{4}x^4(-2y) + \binom{5}{3}x^3(-2y)^2 + \binom{5}{2}x^2(-2y)^3 + \binom{5}{1}x(-2y)^4 + \binom{5}{0}(-2y)^5$$
$$= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$$

15. Evaluate the following sums.

(a) $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = \sum_{r=0}^{n} \binom{n}{r}$

Since $(1 + x)^n = \sum_{r=0}^{n} \binom{n}{r}x^r$, $2^n = \sum_{r=0}^{n} \binom{n}{r}$ by setting $x = 1$.

(b) $\binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n\binom{n}{n} = \sum_{r=0}^{n} (-1)^r\binom{n}{r}$

If instead, we let $x = -1$, $0 = \sum_{r=0}^{n} (-1)^r\binom{n}{r}$.

16. Sum $\sum_{r=0}^{n} r\binom{n}{r}$ by first showing $\sum_{r=0}^{n} r\binom{n}{r} = \sum_{r=0}^{n} (n - r)\binom{n}{r}$ and using 15(a).

By 13(a), $\sum_{r=0}^{n} r\binom{n}{r} = \sum_{r=0}^{n} (n - r)\binom{n}{n-r} = \sum_{r=0}^{n} (n - r)\binom{n}{r}$.

Thus, $2\sum_{r=0}^{n} r\binom{n}{r} = n\sum_{r=0}^{n} \binom{n}{r} = n \cdot 2^n$ by 15(a), and the sum is $n \cdot 2^{n-1}$.

17. If $P_n(x)$ denotes a polynomial of degree $n$ such that $P_n(x) = 2^x$ for $x = 0, 1, 2, \ldots, n$ find $P_n(n + 1)$.

For this problem, it will be convenient to set

$$Q_r(x) = \frac{x(x - 1) \ldots (x - r + 1)}{r!}$$

where $r$ is a non-negative integer. Note for any integer $n \geq r$ that $Q_r(n) = \binom{n}{r}$. Consider the $n$-th degree polynomial

$$P_n(x) = \sum_{r=0}^{n} Q_r(x) = 1 + x + \frac{x(x-1)}{1 \cdot 2} + \dots + \frac{x(x-1)\dots(x-r+1)}{r!}$$

$P_n(0) = \binom{0}{0} = 1,$

$P_n(1) = \binom{1}{0} + \binom{1}{1} = 2,$

$P_n(2) = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 2^2;$

$P_n(x) = 2^x$ for $x = 0, 1, 2, \dots, n$ by Number 15(a), and thus satisfies our requirements.

$$P_n(n+1) = \binom{n+1}{0} + \binom{n+1}{1} + \dots + \binom{n+1}{n}$$

$$= \sum_{r=0}^{n+1} \binom{n+1}{r} - \binom{n+1}{n+1}$$

$$= 2^{n+1} - 1.$$

## Solutions Exercises A3-2b

1. Write the following sums in telescoping form, i.e., in the form $\sum_{k=1}^{n} \{u(k) - u(k-1)\}$ and evaluate

(a) $\sum_{k=1}^{n} k(k+1)$

(b) $\sum_{k=1}^{n} k(2k-1)$

(c) $\sum_{k=1}^{n} 2k(2k+1)$

(d) $\sum_{k=1}^{n} k(k+1)(k+2)$

(e) $\sum_{k=1}^{n} k^3$

(f) $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}$

(g) $\sum_{k=1}^{n} k \cdot k!$

(h) $\sum_{k=1}^{n} r^k$

(a) $\frac{1}{3}\sum_{k=1}^{n}\{k(k+1)(k+2)-(k-1)k(k+1)\} = \frac{n(n+1)(n+2)}{3}$

(b) $k(2k-1) = 2k(k+1) - 3k$. Using (a) and $3k = \frac{3}{2}\{k(k+1)-(k-1)k\}$, the sum is $\frac{2n(n+1)(n+2)}{3} - \frac{3n(n+1)}{2}$.

(c) $2k(2k+1) = 4k(k+1) - 2k$. Using (a) and (b), the sum is $\frac{4n(n+1)(n+2)}{3} - n(n+1)$.

(d) Here, $u(k) = \frac{k(k+1)(k+2)(k+3)}{4}$ and the sum is $\frac{n(n+1)(n+2)(n+3)}{4}$.

(e) $k^3 = k(k+1)(k+2) - 3k(k+1) - k$. Whence

$$u(k) = \frac{k(k+1)(k+2)(k+3)}{4} - \frac{3k(k+1)(k+2)}{3} - \frac{k(k+1)}{2}$$

and the sum is

$$\frac{n(n+1)(n+2)(n+3)}{4} - n(n+1)(n+2) - \frac{n(n+1)}{2}.$$

(f) Here, $u(k) = -\frac{1}{2(k+1)(k+2)}$ and the sum is $\frac{1}{2}\{\frac{1}{1\cdot 2} - \frac{1}{(n+1)(n+2)}\}$.

(g) Here, $u(k) = (k+1)!$ and the sum is $(n+1)! - 1$.

(h) Here, $u(k) = \frac{r^{k+1}}{r-1}$ and the sum is $\frac{r^{n+1}-1}{r-1}$, $r \neq 1$.

2. Using $\sum_{k=1}^{n}\{u(k) - u(k-1)\} = u(n) - u(0)$, establish a short dictionary of summation formulae by considering the following functions $u$:

(a) $(a+kd)(a+(k+1)d)\ \dots\ (a+(k+p)d)$

(b) The reciprocal of (a).

(c) $r^k$

(d) $kr^k$

(e) $k^2r^k$

(f) $k!$

(g) $(k!)^2$

(h) $\arctan k$

(i) $k \sin k$

(a) $(p+1)d\sum_{k=1}^{n}(a+kd)(a+(k+1)d)\ldots(a+(k+p-1)d)$

$$= (a+nd)(a+(n+1)d)\ldots(a+(n+p)d) - a(a+d)\ldots(a+pd).$$

(b) $(p+1)d\sum_{k=1}^{n}\{(a+(k-1)d)(a+kd)\ldots(a+(k+p)d)\}^{-1}$

$$= \{a(a+d)\ldots(a+pd)\}^{-1} - \{(a+nd)(a+(n+1)d)\ldots(a+(k+p)d)\}^{-1}.$$

(c) $(r-1)\sum_{k=1}^{n} r^{k-1} = r^n - 1.$

(d) $(r-1)\sum_{k=1}^{n} kr^{k-1} + \sum_{k=1}^{n} r^{k-1} = nr^n$ or

$$\sum_{k=1}^{n} kr^{k-1} = \frac{n(r-1)r^n - r^n + 1}{(r-1)^2}$$

(e) $(r-1)\sum_{k=1}^{n} k^2r^{k-1} = n^2r^n + \sum_{k=1}^{n} r^{k-1} - 2\sum kr^{k-1}.$

(Now use (c) and (d).)

(f) $\sum_{k=1}^{n}(k-1)!(k-1) = n! - 1.$

(g) $\sum_{k=1}^{n}(k^2-1)\{(k-1)!\}^2 = (n!)^2 - 1$

(h) $\sum_{k=1}^{n}\arctan\frac{1}{k^2-k+1} = \arctan n$

(i) $\frac{1}{2}(\sin\frac{1}{2})\sum_{k=1}^{n} k\cos(k-\frac{1}{2}) = n\sin n - \sum_{k=1}^{n}\sin(k-1$

(Now use Equation 8.)

3. Simplify:

$$\frac{\sin x + \sin 3x + \ldots + \sin((2n-1)x)}{\cos x + \cos 3x + \ldots + \cos((2n-1)x)}.$$

Since

$$\sum_{k=1}^{n} \cos(ak + b - \tfrac{a}{2}) = \cos(b + \tfrac{an}{2}) \frac{\sin \frac{an}{2}}{\sin \frac{a}{2}},$$

$$\sum_{k=1}^{n} \sin x(2k - 1) = \frac{\sin xn \sin xn}{\sin x}$$

by letting $a = -2x$, $b = \frac{\pi}{2}$, and

$$\sum_{k=1}^{n} \cos x(2k - 1) = \frac{\cos xn \sin xn}{\sin x}$$

by letting $a = 2x$, $b = 0$.

Whence,

$$\frac{\sum_{k=1}^{n} \sin x(2k - 1)}{\sum_{k=1}^{n} \cos x(2k - 1)} = \tan xn.$$

4. Another method for summing $\sum P(k)$ ($P$ - a polynomial) can be obtained by using a special case of Number 2a, i.e.,

$$\sum_{k=1}^{n} \{(k+1)(k)(k-1)\ldots(k-r+1) - (k)(k-1)(k-2)\ldots(k-r)\}$$
$$= (n+1)(n)(n-1)\ldots(n-r+1),$$

or

$$\sum_{k=1}^{n} k(k-1)\ldots(k-r+1) = \frac{(n+1)(n)(n-1)\ldots(n-r+1)}{r+1}.$$

First, we show how to represent any polynomial $P(k)$ of $r$-th degree in the form

(i) $P(k) = a_0 + a_1 k + \frac{a_2 k(k-1)}{2!} + \ldots + \frac{a_r k(k-1)\ldots(k-r+1)}{r!}.$

If $k = 0$, then $a_0 = P(0)$; if $k = 1$, then $a_1 = P(1) - P(0)$; if $k = 2$, then $a_2 = P(2) - 2P(1) + P(0)$. In general, it can be shown that

(ii) $a_m = P(m) - \binom{m}{1}P(m-1) + \binom{m}{2}P(m-2) - \ldots + (-1)^m P(0),$

$m = 0, 1, \ldots, r.$

Since both sides of (i) are polynomials of degree $r$ and (i) is satisfied for $m = 0, 1, \ldots, r$, it must be an identity.

Now sum $\sum_{k=1}^{n} P(k)$.

$$\sum_{k=1}^{n} P(k) = a_0 n + \frac{a_1(n+1)(n)}{2!} + \ldots + \frac{a_r(n+1)(n)(n-1)\ldots(n-r+1)}{r!}.$$

5. Using Problem 4, find the following sums:

(a) $\sum_{k=1}^{n} k^2$

$k^2 = a_0 + a_1 k + \frac{a_2 k(k-1)}{2!}$ where $a_0 = 0^2$, $a_1 = 1^2 - 0^2 = 1$,

$a_2 = 2^2 - 2(1) + 0 = 2$. Thus, $k^2 = k + k(k-1)$ and

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)}{2} + \frac{(n+1)(n)(n-1)}{3} = \frac{n(n+1)(2n+1)}{6}.$$

(b) $\sum_{k=1}^{n} k^3 - \left(\sum_{k=1}^{n} k\right)^2$

$k^3 = a_0 + a_1 k + \frac{a_2 k(k-1)}{2!} + \frac{a_3 k(k-1)(k-2)}{3!}$ where $a_0 = 0$,

$a_1 = 1$, $a_2 = 2^3 - 2(1) = 6$, $a_3 = 3^3 - 3(8) + 3(1) = 6$. Thus,

$k^3 = k + 3k(k-1) + k(k-1)(k-2)$ (compare with Number 1e)

and

$$\sum_{k=1}^{n} k^3 = \frac{(n+1)n}{2} + \frac{3(n+1)(n)(n-1)}{3} + \frac{(n+1)(n)(n-1)(n-2)}{4}$$

$$= \frac{n^2(n+1)^2}{4}$$

$\sum_{k=1}^{n} k = \frac{(n+1)(n)}{2}$. Finally, $\sum_{k=1}^{n} k^3 - \left(\sum_{k=1}^{n} k\right)^2 = 0$.

(c) $\sum_{k=1}^{n} k^4$

$$k^4 = a_0 + a_1 k + \dots + \frac{a_4 k(k-1)(k-2)(k-3)}{4!}$$

$a_0 = 0$, $a_1 = 1$, $a_2 = 2^4 - 2(1) = 14$,

$a_3 = 3^4 - 3(2^4) + 3(1) = 36$,

$a_4 = 4!$. Whence,

$$\sum_{k=1}^{n} k^4 = \frac{(n+1)(n)}{2} + \frac{7(n+1)(n)(n-1)}{3} + \frac{6(n+1)(n)(n-1)(n-2)}{4} + \frac{(n+1)(n)(n-1)(n-2)(n-3)}{5}.$$

6. (a) Establish Equation (ii) of Number 4.

Since $a_0, a_1, \dots, a_r$ are defined by the equation

$$P(k) = a_0\binom{k}{0} + a_1\binom{k}{1} + \dots + a_r\binom{k}{r},$$

we have the following $r$ linear equations for the $a_i$'s:

$$P(0) = a_0\binom{0}{0},$$

$$P(1) = a_0\binom{1}{0} + a_1\binom{1}{1},$$

$$P(2) = a_0\binom{2}{0} + a_1\binom{2}{1} + a_2\binom{2}{2},$$

$$\vdots \qquad\qquad \vdots$$

$$P(r) = a_0\binom{r}{0} + a_1\binom{r}{1} + \dots + a_r\binom{r}{r}.$$

Our proof is by mathematical induction. Assume that

(A) $a_n = P(n)\binom{n}{0} - P(n-1)\binom{n}{1} + \ldots + (-1)^n P(0)\binom{n}{n} = \sum_{k=0}^{n} (-1)^k P(n-k)\binom{n}{k}$

is valid for $n = 0, 1, 2, \ldots, m - 1$. We now wish to show that the expression for $a_n$ is also valid for $n = m$. This is equivalent to showing

(B) $P(m) = a_0\binom{m}{0} + a_1\binom{m}{1} + \ldots + a_m\binom{m}{m} = \sum_{j=0}^{m} a_j\binom{m}{j}$

(for the values of $a_n$ given above, $n = 1, 2, \ldots, m$). This will involve manipulations on double series.

$$\sum_{j=0}^{m} a_j\binom{m}{j} = \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j}(-1)^k P(j-k)\binom{j}{k}$$

(by substituting for $a_j$ in (A)).

Now let $k = j - i$. Then,

$$\sum_{j=0}^{m} a_j\binom{m}{j} = \sum_{j=0}^{m} \sum_{i=0}^{j} \binom{m}{j}(-1)^{j-i} P(i)\binom{j}{j-i}.$$

Noting that $\binom{j}{j-i} = \binom{j}{i}$ and interchanging order of summation, we get

$$\sum_{j=0}^{m} a_j\binom{m}{j} = \sum_{i=0}^{m} \sum_{j=i}^{m} \binom{m}{j}(-1)^{j-i} P(i)\binom{j}{i}$$

$$= \sum_{i=0}^{m} P(i) \sum_{j=i}^{m} (-1)^{j-i}\binom{m}{j}\binom{j}{i}.$$

Since $\binom{m}{j}\binom{j}{i} = \binom{m}{i}\binom{m-i}{j-i}$,

$$\sum_{j=i}^{m} (-1)^{j-i}\binom{m}{j}\binom{j}{i} = \sum_{j=i}^{m} (-1)^{j-i}\binom{m}{i}\binom{m-i}{j-i}.$$

Now let $j = i + r$, which reduces the last summation to

$$\binom{m}{i}\sum_{r=0}^{m-r} (-1)^r\binom{m-i}{r} = 0 \text{ if } i \neq m$$
$$= 1 \text{ if } i = m$$

(see Exercises A3-2a, No. 15b).

Finally, $\sum_{j=0}^{m} a_j\binom{m}{j} = P(m)$ which was to be shown.

Since our inductive hypothesis (A) is valid for $n = 0$, it is valid for all $n$.

(b) Show that $a_m$ is zero for $m > r$.

Suppose we wanted the equation $F(x) = a_0\binom{x}{0} + a_1\binom{x}{0} + \ldots + a_m\binom{x}{m}$ (where $m$ is any number $> r$), to be satisfied for $x = 0, 1, 2, \ldots, m$ where $F(x)$ is some given function. By setting $x = 0, 1, 2, \ldots, m$ in turn the $a_i$'s will have to satisfy

$$F(0) = a_0\binom{0}{0},$$

$$F(1) = a_0\binom{1}{0} + a_1\binom{1}{1},$$

$$\vdots \qquad\qquad \vdots$$

$$F(m) = a_0\binom{m}{0} + a_1\binom{m}{1} + \ldots + a_m\binom{m}{m}.$$

It follows (from algebra) that this system of $(m + 1)$ linear equations in $(m + 1)$ unknowns has a unique solution for all $F(x)$. By our inductive argument in part (a), the solution is given as

$$a_n = F(n)\binom{n}{0} - F(n - 1)\binom{n}{1} + \ldots + (-1)^n F(0)\binom{n}{n}$$

for $n = 0, 1, 2, \ldots, m$.

If we now choose $F(x)$ to be the polynomial $P(x)$ of degree $r$ in part (a), then $P(x)$ is identical to

$$a_0\binom{x}{0} + a_1\binom{x}{1} + \ldots + a_r\binom{x}{r}$$

(from Problem 4). It then follows that

$$a_{r+1}\binom{x}{r+1} + a_{r+2}\binom{x}{r+2} + \ldots + a_m\binom{x}{m}$$

vanishes for $x = 0, 1, 2, \ldots, m$. If a polynomial of degree $m$ vanishes for $m + 1$ different values it must identically vanish. Therefore,

$$a_{r+1} = a_{r+2} = \ldots = a_m = 0$$

for all $m > r$.

Teachers Commentary

Appendix 4

FURTHER TECHNIQUES OF INTEGRATION

## A4-1. Substitutions of Circular Functions

An integral of a rational combination of $\sinh x$ and $\cosh x$ can be transformed into an integral of a rational function by a substitution exploiting the analogy between circular and hyperbolic functions. However, it is simpler to recognize that the integrand is a rational function of $e^x$ (See Exercises A4-1, No. 10).

The integration of $\frac{1}{\cos\theta}$ which occurs in Example A4-1d may be accomplished by the substitution $u = \sin\theta$, as follows:

$$\int \frac{d\theta}{\cos\theta} = \int \frac{\cos\theta}{1 - \sin^2\theta}\, d\theta = \int \frac{du}{1 - u^2}$$

$$= \operatorname{argtanh} \sin\theta + C$$

$$= \frac{1}{2} \log \frac{1 + \sin\theta}{1 - \sin\theta} + C.$$

### Solutions Exercises A4-1

1. Integrate the following functions, the numbers $a$ and $b$ being positive.

(a) $\frac{\sqrt{a^2 - x^2}}{x^2}$. Set $x = a\cos\theta$.

$$I = -\int \tan^2\theta\, d\theta = -\int [(1 + \tan^2\theta) - 1]d\theta$$

$$= \theta - \tan\theta + C,$$

$$= \arccos\frac{x}{a} - \frac{\sqrt{a^2 - x^2}}{x} + C.$$

(b) $\dfrac{\sqrt{1+x^2}}{x^4}$.  Set $x = \tan t$.

$$I = \int \frac{\cos t}{\sin^4 t}\,dt = -\frac{1}{3\sin^3 t} + C$$

$$= -\frac{1}{3}\left(\frac{\sqrt{1+x^2}}{x}\right)^3 + C.$$

(Alternatively, set $u = \dfrac{1}{x^2}$ to obtain $I = -\dfrac{1}{2}\displaystyle\int \sqrt{1+u}\,du$.)

(c) $x^2\sqrt{a^2 - x^2}$.  Set $x = a\sin t$.

$$I = a^4 \int \sin^2 t \cos^2 t\,dt = \frac{a^4}{4}\int \sin^2 2t\,dt$$

$$= \frac{a^4}{8}\int (1 - \cos 4t)\,dt = \frac{a^4}{8}\left(t - \frac{1}{4}\sin 4t\right) + C$$

$$= \frac{a^4}{8}\arcsin\frac{x}{a} + \left(\frac{1}{4}x^3 - \frac{a^2}{8}x\right)\sqrt{a^2 - x^2} + C.$$

(d) $\dfrac{1}{x^2\sqrt{x^2 - a^2}}$.  Set $x = a\cosh u$.

$$I = \frac{1}{a^2}\int \frac{du}{\cosh^2 u} = \frac{1}{a^2}\tanh u + C$$

$$= \frac{\sqrt{a^2 - x^2}}{a^2 x} + C.$$

(e) $\dfrac{x}{(x^2 + a^2)\sqrt{x^2 - b^2}}$.  Set $x^2 - b^2 = t^2$.

$$I = \int \frac{dt}{a^2 + b^2 + t^2} = \frac{1}{\sqrt{a^2 + b^2}}\arctan\frac{t}{\sqrt{a^2 + b^2}} + C$$

$$= \frac{1}{\sqrt{a^2 + b^2}}\arctan\sqrt{\frac{x^2 - b^2}{a^2 + b^2}} + C.$$

(f) $\dfrac{1}{(x^2 + a^2)\sqrt{a^2x^2 + 1}}$.  Set $x = \frac{1}{a}\sinh t$.

$$I = \int \frac{a\,dt}{a^4 + \sinh^2 t} = a\int \frac{1}{a^4 + (1 - a^4)\tanh^2 t}\,\frac{1}{\cosh^2 t}\,dt$$

$$= a\int \frac{ds}{a^4 + (1 - a^4)s^2}$$

where $s = \tanh t$.

If $a = 1$, then $I = s + C = \dfrac{x}{\sqrt{x^2 + 1}} + C$.

If $a = 0$, then $I = -\frac{1}{x} + C$.

If $0 < a < 1$, then

$$I = \frac{1}{a\sqrt{1 - a^4}} \arctan \frac{\sqrt{1 - a^4}}{a^2} s + C$$

$$= \frac{1}{a\sqrt{1 - a^4}} \arctan \frac{x}{a}\sqrt{\frac{1 - a^4}{1 + a^2x^2}} + C.$$

If $a > 1$, then for $b = \dfrac{a^2}{\sqrt{a^4 - 1}}$,

$$I = \frac{b^2}{a^3}\int \frac{ds}{b^2 - s^2} = \frac{b}{2a^3}\log\left|\frac{b + s}{b - s}\right| + C$$

$$= \frac{b}{2a^3}\log\left|\frac{b\sqrt{1 + a^2x^2} + ax}{b\sqrt{1 + a^2x^2} - ax}\right|.$$

(g) $\dfrac{x + 2}{\sqrt{m^2 \pm x^2}} = \dfrac{x}{\sqrt{m^2 \pm x^2}} + \dfrac{2}{\sqrt{m^2 \pm x^2}}$.

Plus sign: $I = \sqrt{m^2 + x^2} + 2\,\mathrm{argsinh}\, x + C$.

Minus sign: $I = -\sqrt{m^2 - x^2} + 2\arcsin x + C$.

(h) $x^3\sqrt{(4 - x^2)^5}$.  Set $\sqrt{4 - x^2} = t$.

$$I = \int (t^2 - 4)t^6\,dt = \frac{-(4 - x^2)^{7/2}(8 + 7x^2)}{63} + C.$$

(i) $\dfrac{1}{\sqrt{a^2x - x^2}} = \dfrac{1}{\sqrt{\frac{a^4}{4} - (x - \frac{a^2}{2})^2}}$.

Set $x = \frac{a^2}{2}(\sin\theta + 1)$.

$$I = \arcsin\left(\frac{2x}{a^2} - 1\right) + C.$$

(Alternatively, set $x = a^2\sin^2\psi$ to obtain $I = 2\psi = 2\arcsin\frac{\sqrt{x}}{a}$. This is suggested by the preliminary substitution $x = t^2$.)

(j) $\dfrac{x^2 + ax + b}{x^2 + 1} = \dfrac{(x^2 + 1) + ax + (b - 1)}{x^2 + 1}$.

$$I = x + \frac{a}{2}\log(x^2 + 1) + (b - 1)\arctan x + C.$$

(k) $\sqrt{a^2x + x^2} = \sqrt{(x + \frac{a^2}{2})^2 - \frac{a^4}{4}}$.

Set $x + \frac{a^2}{2} = \frac{a^2}{2}\cosh z$.

$$I = \frac{a^4}{4}\int \sinh^2 z\,dz = \frac{a^4}{8}\int(\cosh 2z - 1)dz$$

$$= (x + \frac{a^2}{2})\sqrt{a^2x + x^2} - \frac{a^4}{8}\operatorname{argcosh}\left(\frac{2x}{a^2} + 1\right) + C.$$

2. Let $R(x,y)$ denote a rational function in $x$ and $y$. Reduce the following integrals to integrals of rational functions.

(a) $\int R(x, \sqrt{ax + b})dx$, $a \neq 0$. Set $\sqrt{ax + b} = t$.

$$I = \frac{2}{a}\int R\left(\frac{t^2 - b}{a}, t\right)t\,dt.$$

(b) $\int R\left(x, \sqrt[n]{\frac{ax + b}{cx + d}}\right)dx$, $n$ - an integer, $ad - bc \neq 0$.

Set $\sqrt[n]{\frac{ax + b}{cx + d}} = t$.

$$I = \int n(ad - bc)\frac{t^{n-1}}{(a - ct^n)^2}R\left\{\frac{dt^n - b}{a - ct^n}, t\right\}dt.$$

3. Using the result of Number 2, integrate $\dfrac{x}{\sqrt{ax+b}+(\sqrt{ax+b})^3}$.

$$I = \frac{2}{a^2}\int \frac{(t^2 - b)t}{t + t^3}\,dt = \frac{2}{a^2}\int \left(1 - \frac{1+b}{1+t^2}\right)dt$$

$$= \frac{2}{a^2}\left[\sqrt{ax+b} - (1+b)\arctan\sqrt{ax+b}\right] + C.$$

4. Reduce to rational form, $\displaystyle\int \frac{dx}{\sqrt{\frac{1-x}{1+x}} + \sqrt[4]{\frac{1-x}{1+x}}}$.

Use the method of Number 2(b).

$$I = -8\int \frac{t^2\,dt}{(1+t)(1+t^4)^2}.$$

5. Express as elementary functions,

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2+1} + \sqrt{x^2-1}}$.

First note that $\dfrac{1}{\sqrt{x^2+1}+\sqrt{x^2-1}} = \dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{2}$.

$$I = \frac{x}{4}\left(\sqrt{x^2+1} - \sqrt{x^2-1}\right) + \frac{1}{4}\log\left|(x+\sqrt{x^2+1})(x+\sqrt{x^2-1})\right| + C.$$

(b) $\displaystyle\int \frac{dx}{1+\sin x} = \int \frac{1-\sin x}{\cos^2 x}\,dx = \tan x - \frac{1}{\cos x} + C.$

(c) $\displaystyle\int \frac{dx}{1-\cos 2x} = \int \frac{1+\cos 2x}{\sin^2 2x}\,dx = -\frac{1}{2}\left(\cot 2x + \frac{1}{\sin 2x}\right) + C.$

(d) $\displaystyle\int \frac{dx}{x\sqrt[4]{1 + x^4}}$. Set $t = \sqrt[4]{1 + x^4}$; then

$$I = \int \frac{t^2}{t^4 - 1}\,dt = \frac{1}{2}\int \frac{(t^2 + 1) + (t^2 - 1)}{t^4 - 1}\,dt$$

$$= \frac{1}{2}\int \left[\frac{1}{t^2 - 1} + \frac{1}{t^2 + 1}\right] dt$$

$$= \frac{1}{4}\log\left|\frac{t - 1}{t + 1}\right| + \frac{1}{2}\arctan t + C$$

$$= \frac{1}{4}\log\frac{\sqrt[4]{1 + x^4} - 1}{\sqrt[4]{1 + x^4} + 1} + \frac{1}{2}\arctan\sqrt[4]{1 + x^4} + C.$$

(e) $\displaystyle\int \frac{dx}{\sqrt[4]{1 + x^4}}$. Set $x = \frac{1}{u}$; then

$$I = -\int \frac{du}{u\sqrt[4]{1 + u^4}}$$

and the problem is reduced to (d).

6. (a) The integral $\displaystyle\int \frac{P(x)}{\sqrt{ax^2 + 2bx + c}}\,dx$ where $P(x)$ is a polynomial of degree $n$ and $a \neq 0$ can be reduced to a rational trigonometric form as described in the text. It can also be reduced to the integration of $\displaystyle\frac{1}{\sqrt{ax^2 + 2bx + c}}$; namely, for some polynomial $Q$ of degree $n - 1$ and constant $k$,

$$\frac{P(x)}{\sqrt{ax^2 + 2bx + c}} = D_x\{Q(x)\sqrt{ax^2 + 2bx + c}\} + \frac{k}{\sqrt{ax^2 + 2bx + c}}.$$

Show how to find $Q$ and $k$.

Since

$$D_x(Q(x)\sqrt{ax^2 + 2bx + c}) = \frac{Q'(x)(ax^2 + 2bx + c) + Q(x)(ax + b)}{\sqrt{ax^2 + 2bx + c}},$$

the polynomial $Q$ and the constant $k$ must satisfy

(1) $P(x) \equiv Q(x)(ax + b) + Q'(x)(ax^2 + 2bx + c) + k.$

The constant $k$ and the coefficients of $Q(x)$ can be found from

the coefficients of $P(x)$ as follows. Set $P(x) = \sum_{\nu=0}^{n} p_\nu x^{n-\nu}$ and $Q(x) = \sum_{\nu=0}^{n-1} q_\nu x^{n-1-\nu}$. Equate coefficients of like powers on the right and left in (1) beginning with the coefficients of $x^n$:

$$p_0' = aq_0 + a(n-1)q_0$$

to obtain

$$q_0 = \frac{p_0}{an}.$$

Thus the leading coefficient of $Q(x)$ is determined. Next, $p_1 = (aq_1 + bq_0) + [a(n-2)q_1 + 2b(n-1)q_0]$, or $a(n-1)q_1 = p_1 - b(2n-1)q_0$. Similarly all succeeding coefficients are determined step-by-step from the preceding ones:

$$a(n-\nu)q_\nu = p_\nu - b(2n-2\nu-1)q_{\nu-1} - c(n-\nu+1)q_{\nu-2},$$

for $\nu = 2, \ldots, n-1$. Finally, for the constant $k$,

$$k = p_0 - bq_0 - cq_1.$$

(b) Using (a), integrate $\dfrac{t^5 - t^3 + t}{\sqrt{1-t^2}}$.

Set $Q(t) = q_0t^4 + q_1t^3 + q_2t^2 + q_3t + q_4$ and $P(t) = t^5 - t^3 + t$.

$$D(Q(t)\sqrt{1-t^2}) = \frac{1}{\sqrt{1-t^2}}[-5q_0t^5 - 4q_1t^4 + (4q_0 - 3q_2)t^3$$
$$+ (3q_1 - 2q_3)t^2 + (2q_2 - q_4)t + q_3].$$

Now solve for the coefficients in succession to obtain

$$q_0 = -\frac{1}{5},\ q_1 = 0,\ q_2 = \frac{1}{15},\ q_3 = 0;$$
$$q_4 = -\frac{13}{15},\ k = 0.$$

Consequently,

$$I = \int \frac{t^5 - t^3 + t}{\sqrt{1-t^2}}\,dt = -\frac{1}{15}(3t^4 - t^2 + 13)\sqrt{1-t^2} + C.$$

(c) Find the integral of (b) by using trigonometric substitutions, and compare the merits of the two methods.

Set $t = \sin\theta$. Then

$$I = \int (\sin^4\theta - \sin^2\theta + 1)\sin\theta\, d\theta$$

$$= \int [(1 - \cos^2\theta)^2 - (1 - \cos^2\theta) + 1]\sin\theta\, d\theta$$

$$= \int [(\cos^4\theta - \cos^2\theta + 1]\sin\theta\, d\theta$$

$$= -\frac{1}{5}\cos^5\theta + \frac{1}{3}\cos^3\theta - \cos\theta + C$$

$$= -\frac{1}{15}(3t^4 - t^3 + 13)\sqrt{1 - t^2} + C.$$

which is the result obtained in (b). If even powers appeared in $P(t)$ the work in (c) would be more complicated while the work in (b) would not change greatly. Furthermore the method of (a) eliminates the repetitive use of the binomial theorem when $P$ is of high degree.

7. Integrate

(a) $\dfrac{1}{\sin x}$

Use $x = 2\arctan t$ to obtain

$$I = \int \frac{dt}{t} = \log\left|\tan\frac{x}{2}\right| + C.$$

(b) $\dfrac{1}{\cos x}$ (by a method other than that of Example A4-1d).

Use $x = 2\arctan t$ to obtain

$$I = \int \frac{dt}{1 - t^2} = \frac{1}{2}\int \left[\frac{1}{1 - t} + \frac{1}{1 + t}\right]dt$$

$$= \frac{1}{2}\log\left|\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right| + C.$$

Alternatively, from $\cos x = \sin u$, where $u = x + \frac{\pi}{2}$, obtain from the solution of Part (a),

$$I = \int \frac{du}{\sin u} = \log\left|\tan \frac{1}{2}\left(x + \frac{\pi}{2}\right)\right| + C.$$

## Integration by Parts

**THEOREM A4-2.** When we say "$\phi$ and $\psi$ are inverses" we mean $\phi$ is the inverse of $\psi$ and $\psi$ is the inverse of $\phi$. (See, also, Exercises A4-2, No. 2.)

Example A4-2b. It might be appropriate to first consider $\int \arcsin x\,dx$. We observe that

$$\arcsin x = (\arcsin x)(1) = (\arcsin x)\frac{dx}{dx}.$$

Thus $\arcsin x = u\frac{dv}{dx}$ where $u = \arcsin x$ and $v = x$. Integrating by parts we get

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\arcsin x + \sqrt{1-x^2} + C.$$

It should be noted that the discussion in the text establishes the integrability of $x^n \arcsin x$.

### Solutions Exercises A4-2

1. Integrate the following functions.

In these solutions the integral of the problem is written $I = \int u\,dv$.

(a) $x \sin 3x$

$$I = -x\frac{\cos 3x}{3} + \frac{\sin 3x}{9} + C.$$

(b) $x \cdot 5^x$

$$I = \frac{x \cdot 5^x}{\log 5} - \frac{5^x}{(\log 5)^2} + C.$$

(c) $x^3 e^{-2x}$

Set $dv = e^{-2x}dx$ and integrate by parts three times.

$$I = -e^{-2x}\left(\frac{x^3}{2} + \frac{3x^2}{4} + \frac{3x}{4} + \frac{3}{8}\right) + C.$$

(d) $\sqrt{x}\log ax$

Set $u = \log ax$, $dv = \sqrt{x}\,dx$.

$$I = \frac{2}{3}x^{3/2}\left(\log ax - \frac{2}{3}\right) + C.$$

Alternatively, set $t = ax$ and use the result of Example A4-2c.

(e) $\log^2 bx$ 	Set $u = \log^2 bx$.

$$I = x\log^2 bx - 2\int \log bx\,dx + C$$
$$= x\log^2 bx - 2x(\log bx - 1) + C$$

where the integration of Example 10-4a is used at the end.

(f) $\log^3 x$ 	Set $u = \log^3 x$. Apply (e).

$$I = x(\log^3 x - 3\log^2 x + 6\log x - 6) + C.$$

(g) $\arccos 7x$ 	Set $u = \arccos 7x$. Then

$$I = x\arccos 7x - \frac{\sqrt{1-49x^2}}{7} + C.$$

(h) $\operatorname{argsinh} ax$ 	Set $u = \operatorname{argsinh} ax$.

$$I = x\operatorname{argsinh} ax - \frac{1}{a}\sqrt{1+a^2x^2} + C.$$

(i) $\operatorname{argtanh} bx$ 	Set $u = \operatorname{argtanh} bx$.

$$I = x\operatorname{argtanh} bx + \frac{1}{b}\log(1 - b^2x^2) + C.$$

(j) $\operatorname{argtanh}\sqrt{bx}$ 	Set $u = \operatorname{argtanh}\sqrt{bx}$.

$$I = x\operatorname{argtanh}\sqrt{bx} - \frac{1}{2}\int \frac{\sqrt{bx}}{1-bx}\,dx$$
$$= x\operatorname{argtanh}\sqrt{bx} - \frac{1}{b}\int \frac{t^2}{1-t^2}\,dt$$

where $t = \sqrt{bx}$. From $\frac{t^2}{1-t^2} = \frac{1}{1-t^2} - 1$, obtain

$$I = \left(x - \frac{1}{b}\right)\operatorname{argtanh}\sqrt{bx} + \sqrt{\frac{x}{b}} + C.$$

(k) $\arctan \sqrt[3]{x}$ Set $u = \arctan \sqrt[3]{x}$.

$$x \arctan \sqrt[3]{x} - \frac{1}{3}\int \frac{x^{1/3}}{1 + x^{2/3}}\, dx .$$

Set $x = z^3$, then

$$\frac{1}{3}\int \frac{x^{1/3}}{1 + x^{2/3}}\, dx = \int \frac{z^3}{1 + z^2}\, dz$$

$$= \frac{1}{2}\int \frac{t}{1 + t}\, dt \quad (\text{where } t = z^2)$$

$$= \frac{1}{2}\int \left(1 - \frac{1}{1 + t}\right) dt$$

$$= \frac{1}{2}(t - \log|1 + t|) .$$

Consequently,

$$I = x \arctan \sqrt[3]{x} - \frac{x^{2/3}}{2} - \log(x^{2/3} + 1) + C .$$

($\ell$) $x \arctan x$ Set $u = \arctan x$, $dv = x\, dx$.

$$I = \frac{x^2}{2} \arctan x - \frac{1}{2}\int \frac{x^2}{1 + x^2}\, dx .$$

From $\dfrac{x^2}{1 + x^2} = 1 - \dfrac{1}{1 + x^2}$, obtain

$$I = \frac{1}{2}\{(x^2 + 1)\arctan x - x\} + C .$$

(m) $\dfrac{\arccos \frac{x}{m}}{\sqrt{x + m}}$ . Set $u = \arccos \frac{x}{m}$, $v = 2\sqrt{x + m}$.

$$I = 2\sqrt{x + m} \arccos \frac{x}{m} + 2\int \frac{\sqrt{x + m}}{\sqrt{m^2 - x^2}}\, dx.$$

$$= 2\sqrt{x + m} \arccos \frac{x}{m} + 2\int \frac{dx}{\sqrt{m - x}}$$

$$= 2\sqrt{x + m} \arccos \frac{x}{m} - 4\sqrt{m - x} + C .$$

(n) $x \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$I = \frac{x^2}{4} - \frac{1}{2}\int x \cos 2x \, dx.$$

Set $u = x$, $dv = \cos 2x \, dx$, then

$$I = \frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C.$$

(o) $x^2 \sin x$ Set $u = x^2$, $v = -\cos x$.

$$I = -x^2 \cos x + \int 2x \cos x \, dx.$$

Now set $u = x$, $v = \sin x$.

$$\int 2x \cos x \, dx = 2x \sin x - 2\int \sin x \, dx;$$

whence,

$$I = -(x^2 + 2)\cos x + 2x \sin x + C.$$

(p) $x^2 \arcsin ax$, Set $u = \arcsin ax$, $v = \frac{x^3}{3}$.

$$I = \frac{x^3}{3} \arcsin ax - \frac{a}{3}\int \frac{x^3}{\sqrt{1 - a^2x^2}} \, dx.$$

Now set $z = \sqrt{1 - a^2x^2}$, $x^2 = \frac{1 - z^2}{a^2}$, $x\,dx = -\frac{z\,dz}{a^2}$. Then

$$\int \frac{x^3}{\sqrt{1 - a^2x^2}} \, dx = -\frac{1}{a^4}\int (1 - z^2)\,dz;$$

whence

$$I = \frac{x^3}{3} \arcsin ax - \frac{\sqrt{1 - a^2x^2}}{3a^3} + \frac{(\sqrt{1 - a^2x^2})^3}{9a^3} + C.$$

(q) $\cos^3 2x$ Follow the method of Example A4-2g.

More simply, note that

$$\cos^3 2x = (1 - \sin^2 2x)\cos 2x$$

$$I = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6} + C.$$

(r) $\sin^5 x$ Either follow the method of Example A4-2g or use

$$\sin^5 x = (1 - \cos^2 x)^2 \sin x.$$

$$I = -\frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x + C.$$

(s) $\sin(\log ax)$ Set $u = \sin(\log ax)$.

$$I = x\sin(\log ax) - \int a\cos(\log ax)dx.$$

Similarly,

$$\int \cos(\log ax)dx = x\cos(\log ax) + aI;$$

whence,

$$I = x\sin(\log ax) - ax\cos(\log ax) - a^2 I$$

and

$$I = \frac{x}{1 + a^2}\{\sin(\log ax) - a\cos(\log ax)\} + C.$$

(t) $x\tan^2 x$ Set $u = x$, $v = \tan x - x$.

$$I = x\tan x - x^2 - \int(\tan x - x)dx$$

$$= x\tan x + \log|\cos x| - \frac{x^2}{2} + C.$$

(u) $(\arcsin x)^2$ Substitute $x = \sin t$ to obtain $I = \int t^2 \cos t\, dt$ and integrate by parts twice. Alternatively, set $u = (\arcsin x)^2$ to obtain

$$I = x(\arcsin x)^2 - 2\int \frac{x}{\sqrt{1 - x^2}} \arcsin x\, dx.$$

Repeat, to obtain

$$\int \frac{x}{\sqrt{1 - x^2}} \arcsin x\, dx = -\sqrt{1 - x^2} \arcsin x + \int dx.$$

$$I = x(\arcsin x)^2 + 2\sqrt{1 - x^2} \arcsin x - 2x + C.$$

(v) $\sin ax \cos bx$, $(a^2 \neq b^2)$. Set $u = \sin ax$, $v = \frac{\sin bx}{b}$.

$$I = \frac{\sin ax \sin bx}{b} - \frac{a}{b}\int \cos ax \sin bx\, dx.$$

Now set $u = \cos ax$, $v = -\frac{\cos bx}{b}$.

$$\int \cos ax \sin bx\, dx = -\frac{\cos ax \cos bx}{b} - \frac{a}{b} I.$$

Consequently,

$$I = \frac{\sin ax \sin bx}{b} + \frac{a}{b^2} \cos ax \cos bx + \frac{a^2}{b^2} I;$$

whence

$$I = \frac{1}{b^2 - a^2} [a \cos ax \cos bx + b \sin ax \sin bx].$$

More simply, note that

$$\sin ax \cos bx = \frac{1}{2}[\sin(a + b)x + \sin(a - b)x].$$

Hence

$$I = -\frac{1}{2}\left[\frac{\cos(a + b)x}{a + b} + \frac{\cos(a - b)x}{a - b}\right] + C.$$

2. Support the geometrical interpretation of integration by parts by showing for $u = f(x)$ and $v = g(x)$ where $f$ and $g$ have inverses, that $u = \phi(v)$ and $v = \psi(u)$ where $\phi$ and $\psi$ are inverse functions.

Let $F$ be the inverse of $f$, $G$ the inverse of $g$. The functions

$$\phi = fG : v \to fG(v) = f(x) = u$$
$$\psi = gF : u \to gF(u) = g(x) = v$$

are inverses.

3. Verify as alleged after Example A4-2b that the method of the example does demonstrate the reducibility of $\int x^n f(x)dx$ to the integral of a rational function if $f$ is any inverse circular or hyperbolic function, or if $f$ is the logarithmic function.

Set $u = f(x)$, $v = \dfrac{x^{n+1}}{n+1}$. Let $F$ be the inverse of $F$. Then

$$v = \frac{F(u)^{n+1}}{n+1}.$$

If $F$ is a trigonometric or hyperbolic function, then $\int v\, du$ is reducible to the integral of a rational function by Theorem A4-1b or Exercises A4-1, Number 10.

If $u = \log x$, then $F(u) = e^u$. Here,

$$\int v\, du = \int \frac{e^{(n+1)u}}{n+1}\, du = \frac{e^{(n+1)u}}{(n+1)^2} + C$$
$$= \frac{x^{n+1}}{(n+1)^2} + C.$$

Thus, explicitly,

$$\int x^n \log x\, dx = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right) + C.$$

4. Establish recurrence relations for each of the following (in each case $m$ and $n$ are positive integers).

(a) $\int \sin^n x\, dx$

Proceed as in Example A4-2g. Otherwise use the result of the example:

$$I_n = \int \cos^n(x - \tfrac{\pi}{2})dx$$
$$= -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}.$$

(b) $\int x^m \log^n x\, dx$ Set $u = \log^n x$, $v = \dfrac{x^{m+1}}{m+1}$.

$$I_{n,m} = \frac{x^{m+1}}{m+1}\log^n x - \frac{n}{m+1} I_{n-1,m}.$$
$$I_{0,m} = \frac{x^{m+1}}{m+1}.$$

(c) $\int \sin^m x \cos^n x\, dx$ Set $u = \cos^{n-1} x$, $v = \dfrac{\sin^{m+1} x}{m+1}$.

$$I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2}x \cos^{n-2}x\, dx.$$

But, since $\sin^{m+2}x = (1 - \cos^2 x)\sin^m x$,

$$I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+1} + \frac{n-1}{m+1}(I_{m,n-2} - I_{m,n});$$

whence,

$$I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}.$$

Note that $I_{m,0}$ is given by (a).

It is also possible to reduce first $m$ then $n$. Instead of proceeding by the given method, use

$$\sin^m x \cos^n x = (-1)^m \cos^m (x - \tfrac{\pi}{2}) \sin^m (x - \tfrac{\pi}{2})$$

to obtain by the preceding result

$$I_{n,m} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}.$$

(d) $\int x^n \arctan x \, dx$ Set $u = \arctan x$, $v = \frac{x^{n+1}}{n+1}$; Then

(1) $$I_n = \frac{x^{n+1}}{n+1} \arctan x - \frac{1}{n+1}\int \frac{x^{n+1}}{1+x^2}\, dx.$$

But

(2) $$\frac{x^{n+1}}{1+x^2} = \frac{(1+x^2)x^{n-1} - x^{n-1}}{1+x^2}.$$

Insert this in (1) to obtain

$$I_n = \frac{x^{n+1}}{n+1} \arctan x - \frac{x^n}{n(n+1)} + \frac{1}{n+1}\int \frac{x^{n-1}}{1+x^2}\, dx$$

where the integral is given by (1) in terms of $I_{n-2}$. From this,

$$I_n = \frac{x^{n-1}}{n+1}\{(1+x^2)\arctan x - \frac{x}{n}\} - \frac{n-1}{n+1} I_{n-2}$$

Alternatively, reduce the integral in (1) by (2)

$$J_{n+1} = \int \frac{x^{n+1}}{1+x^2}\, dx = \frac{x^n}{n} - J_{n-1}$$

$$= \frac{x^n}{n} - \frac{x^{n-2}}{n-2} + \frac{x^{n-4}}{n-4} - \cdots$$

If $n$ is odd then the sum terminates with $(-1)^{\frac{n-1}{2}} J_1$ where $J_1 = \int \frac{x}{1+x^2}\, dx = \frac{1}{2}\log(1+x^2)$. If $n$ is even then the sum terminates with $(-1)^{n/2} J_0$ where $J_0 = \int \frac{dx}{1+x^2} = \arctan x$.

(e) $\int x^n \operatorname{argsinh} x \, dx$ First substitute $x = \sinh t$ to obtain

$$I_n = \int t \sinh^n t \cosh t \, dt.$$

Now set $u = t$, $v = \frac{\sinh^{n+1} t}{n+1}$; then

(1) $$I_n = t\,\frac{\sinh^{n+1} t}{n+1} - \frac{1}{n+1}\int \sinh^{n+1} t \, dt.$$

Then proceed as in Example A4-2g to obtain

$$J_{n+1} = \int \frac{\sinh^{n+1} t}{n + 1}\,dt$$

$$= \frac{\sinh^n t \cosh t}{(n + 1)^2} - \frac{n(n - 1)}{(n + 1)^2} J_{n-1} .$$

But, from (1)

$$J_{n-1} = \frac{t \sinh^{n-1} t}{n - 1} - I_{n-2} .$$

Combine these results to obtain

$$I_n = \frac{t \sinh^{n+1} t}{n + 1} - \frac{\sinh^n t \cosh t}{(n + 1)^2} + \frac{n}{(n+1)^2} t \sinh^{n-1} t - \frac{n}{(n+1)^2} I_{n-2} ;$$

$$= \frac{x^{n-1}}{n+1}\left(x^2 + \frac{n}{n+1}\right) \operatorname{argsinh} x - \frac{x^n\sqrt{1 + x^2}}{(n + 1)^2} - \frac{n}{(n - 1)^2} I_{n-2} .$$

(f) $\int x^n \operatorname{argtanh} x\,dx$ Proceed as in (d):

$$I = \frac{x^{n+1}}{n + 1} \operatorname{argtanh} x - \frac{1}{n + 1} J_{n+1} ,$$

where

$$J_{n+1} = \int \frac{x^{n+1}}{1 - x^2}\,dx = \int \frac{x^{n-1} - (1 - x^2)x^{n-1}}{1 - x^2}$$

$$= J_{n-1} - \frac{x^n}{n} .$$

(g) $\int x^n e^{ax}\,dx$ Set $u = x^n, \quad v = \frac{e^{ax}}{a}$.

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n - 1}{a} I_{n-1} .$$

(h) $\int x^n \arcsin x\,dx$ Proceed as in (e).

$$I_n = \frac{x^{n-1}}{n + 1} \left(x^2 - \frac{n}{n + 1}\right)\arcsin x + \frac{x^n\sqrt{1 - x^2}}{(n + 1)^2} + \frac{n(n - 1)}{(n + 1)^2} I_{n-2} .$$

(i) $\int \frac{1}{\sin^n x}\,dx$ Set $u = \frac{1}{\sin^{n-2} x}$, $v = -\frac{\cos x}{\sin x}$.

Then $$I_n = \frac{\cos x}{\sin^{n-1} x} - (n-2)\int \frac{\cos^2 x}{\sin^n x}\,dx.$$

Set $\cos^2 x = 1 - \sin^2 x$ in the last integral to obtain

$$I_n = -\frac{1}{n-1}\,\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2}.$$

(j) $\int \frac{e^x}{x^n}\,dx$ Set $u = e^x$, $dv = \frac{1}{x^n}\,dx$.

$$I_n = -\frac{1}{n-1}\,\frac{e^x}{x^n} + \frac{1}{n-1} I_{n-1}.$$

(k) $\int x^n \cos x\,dx$ Set $u = x^n$, $v = \sin x$.

$$I_n = x^n \sin x - n\int x^{n-1} \sin x\,dx.$$

Now set $u = x^{n-1}$, $v = -\cos x$.

$$\int x^{n-1} \sin x\,dx = -x^{n-1} \cos x + (n-1)\int x^{n-2} \cos x\,dx.$$

Consequently,

$$I_n = x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}.$$

For $n$ even, the expansion of $I_n$ terminates with $I_0 = \sin x + C$; for $n$ odd, the expansion ends with $I_1 = x \sin x + \cos x + C$.

A4-3. Integration of Rational Functions

The integral of a rational function is an elementary function, but the computation of such an integral may involve considerable labor depending on the complexity of the function at hand.

Every polynomial with real coefficients has a unique factorization of the form given by Equation (4), but to obtain this form one must first find the zeros of Q. The polynomials which appear in exercises in various textbooks are manufactured artifically for purposes of illustration: they are either given in a factored form, or have zeros which can be found easily. In problems arising in applications this is often not the case.

The method of equated coefficients (Example A4-3c) is, of course, applicable where the roots of Q are all real of multiplicity 1 (as in (7)).

In Example A4-3d since the integrand may be decomposed into the sum of rational functions $\frac{a}{x}$, $\frac{b}{x^2}$, $\frac{cx}{x^2+4}$, $\frac{d}{(\frac{x}{2})^2+1}$ we know that the integral must be of the form stated.

Example TCA4-3a. Consider $I = \int \frac{dx}{x^3(x-1)}$. Note that

$$\frac{1}{x^3(x-1)} = \frac{(1-x)(1+x+x^2)+x^3}{x^3(x-1)} = \frac{1}{x-1} - \frac{1}{x^3} - \frac{1}{x^2} - \frac{1}{x}.$$

Thus we composed the integrand into the sum of simpler rational functions which may be integrated at sight:

$$I = \log|x-1| - \log|x| + \frac{1}{x} + \frac{1}{2x^2} + C.$$

Example TCA4-3b. The integral $I = \int \frac{2x}{x^2+x+1}\,dx$ is computed by decomposing the integrand into a sum of derivatives of two known functions. We have

$$\frac{2x}{x^2+x+1} = \frac{2x+1}{x^2+x+1} - \frac{1}{x^2+x+1}$$

$$= D\log(x^2+x+1) - D\left(\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}\right).$$

Thus $I = \log(x^2+x+1) - \frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C.$

## Solutions Exercises A4-3

1. Integrate the following.

Parts (a) to (k) are simple and an ad hoc approach is probably simpler and quicker than a straight forward application of theory.

(a) $\dfrac{x + 2}{x^2 + 3x + 1}$

Set $x^2 + 3x + 1 = (x - a)(x - b)$ where

$$a = \frac{-3 + \sqrt{5}}{2} \quad \text{and} \quad b = \frac{-3 - \sqrt{5}}{2}.$$

$$\frac{x + 2}{(x - a)(x - b)} = \frac{(x - a) + 2 + a}{(x - a)(x - b)} = \frac{1}{x - b} + \frac{2 + a}{b - a}\left(\frac{1}{x - b} - \frac{1}{x - a}\right)$$

$$= \frac{2 + b}{b - a}\,\frac{1}{x - b} - \frac{2 + a}{b - a}\,\frac{1}{x - a},$$

where

$$\frac{2 + b}{b - a} = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right) \qquad \frac{2 + a}{b - a} = -\frac{1}{2}\left(1 + \frac{1}{\sqrt{5}}\right).$$

$$I = \frac{1}{2}\left(1 - \frac{\sqrt{5}}{5}\right)\log\left|x + \frac{3 + \sqrt{5}}{2}\right| + \frac{1}{2}\left(1 + \frac{\sqrt{5}}{5}\right)\log\left|x + \frac{3 - \sqrt{5}}{2}\right| + C.$$

(b) $\dfrac{x^3}{(x^2 + 3x - 10)}$

By long division,

$$\frac{x^3}{x^2 + 3x - 10} = x - 3 + \frac{19x - 30}{x^2 + 3x - 10}.$$

But

$$\frac{19x - 30}{x^2 + 3x - 10} = \frac{19x - 30}{(x + 5)(x - 2)}$$

$$= \frac{8}{7(x - 2)} + \frac{125}{7(x + 5)}.$$

Consequently,

$$I = \frac{x^2}{2} - 3x + \frac{8}{7}\log|x - 2| + \frac{125}{7}\log|x + 5| + C.$$

(c) $\dfrac{x^3}{x^2 + 2ax + b^2}$, $(b > |a|)$

$$\frac{x^3}{x^2 + 2ax + b^2} = x - 2a + \frac{(2a^2 - \frac{b^2}{2})[2(x + a)] + (3ab^2 - 4a^3)}{x^2 + 2ax + b^2}$$

$$I = \frac{x^2}{2} - 2ax + (2a^2 - \frac{b^2}{2})\log(x^2 + 2ax + b^2) + \frac{3ab^2 - 4a^3}{\sqrt{b^2 - a^2}} \arctan \frac{x + a}{\sqrt{b^2 - a^2}} + C.$$

(d) $\dfrac{x^2 + \alpha x + \beta}{(x - a)(x - b)}$, (consider the cases $a \neq b$ and $a = b$).

If $a \neq b$,

$$I = x + \frac{a^2 + \alpha a + \beta}{a - b} \log |x - a| + \frac{b^2 + \alpha b + \beta}{b - a} \log |x - b| + C.$$

If $a = b$,

$$I = x + (\alpha + 2a)\log(x - a) - \frac{a^2 + \alpha a + \beta}{x - a} + C.$$

(e) $\dfrac{x^2}{(x - a)(x - b)(x - c)}$, ($a$, $b$, $c$ distinct).

$$I = \frac{a^2}{(a - b)(a - c)} \log |x - a| + \frac{b^2}{(b - a)(b - c)} \log |x - b|$$

$$+ \frac{c^2}{(c - a)(c - b)} \log |x - c| + C.$$

(f) $\dfrac{x^3 + 1}{x^3 - 1} = 1 + \dfrac{2}{(x - 1)(x^2 + x + 1)} = 1 + \dfrac{2}{3(x - 1)} - \dfrac{2x + 4}{3(x^2 + x + 1)}$

$$I = x + \frac{2}{3}\log|x - 1| - \frac{1}{3}\log(x^2 + x + 1) - \frac{2}{\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}} + C$$

(g) $\dfrac{1}{x^3 + a^3} = \dfrac{1}{(x + a)(x^2 - ax + a^2)} = \dfrac{1}{3a^2(x + a)} - \dfrac{x - 2a}{3a^2(x^2 - ax + a^2)}$

$$I = \frac{1}{3a^2} \log|x + a| - \frac{1}{6a^2} \log(x^2 - ax + a^2) - \frac{5\sqrt{3}}{9a^2} \arctan \frac{2x - a}{a\sqrt{3}} + C$$

(h) $\dfrac{(x+2)^2}{x(x-1)^2} = \dfrac{9}{(x-1)^2} - \dfrac{3}{x-1} + \dfrac{4}{x}$

$$I = -\frac{9}{x-1} - 3\log|x-1| + 4\log|x| + C$$

(i) $\dfrac{1}{x^4-1} = \dfrac{1}{4(x-1)} - \dfrac{1}{4(x+1)} - \dfrac{1}{2(x^2+1)}$.

$$I = \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\arctan x + C$$

(j) $\dfrac{x^2}{x^4-1} = \dfrac{x^2-1}{x^4+1} + \dfrac{1}{x^4-1} = \dfrac{1}{4(x-1)} - \dfrac{1}{4(x+1)} + \dfrac{1}{2(x^2+1)}$

(see (i)).

$$I = \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\arctan x + C.$$

(k) $\dfrac{1}{x^4+x^6} = \dfrac{1}{x^4(1+x^2)} = \dfrac{1}{x^4} - \dfrac{1}{x^2} + \dfrac{1}{1+x^2}$.

$$I = -\frac{1}{3x^3} + \frac{1}{x} + \arctan x + C$$

($\ell$) $\dfrac{x^4}{x^4+1} = 1 - \dfrac{1}{x^4+1}$

Since $x^4 + 1 > 0$ for all $x$, $x^4 + 1$ cannot have linear factors. Set

$$(x^4+1) = (x^2+ax+b)(x^2+cx+d).$$

Equate coefficients of $x^3$ to obtain $c = -a$; of $x$, to obtain $b = d$ ($a = 0$ is not possible), of $x^0$ to obtain $b = \pm 1$, of $x^2$, to obtain $a^2 = \pm 2$, hence only $b = 1$ is possible and $a = \sqrt{2}$.

$$(x^4+1) = (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1)$$

Set

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2+x\sqrt{2}+1} + \frac{Cx+D}{x^2-x\sqrt{2}+1}.$$

Use the method of undetermined coefficients to obtain $A = -C = \dfrac{1}{2\sqrt{2}}$, $B = D = \dfrac{1}{2}$; whence,

$$\frac{1}{x^4+1} = \frac{1}{4\sqrt{2}}\frac{2x+\sqrt{2}}{x^2+x\sqrt{2}+1} - \frac{1}{4\sqrt{2}}\frac{2x-\sqrt{2}}{x^2-x\sqrt{2}+1}$$

$$+ \frac{1}{4}\frac{1}{x^2+x\sqrt{2}+1} - \frac{1}{4}\frac{1}{x^2-x\sqrt{2}+1}.$$

Consequently,

$$I = \frac{1}{4\sqrt{2}} \log \left|\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}\right|$$

$$- \frac{1}{2\sqrt{2}}\{\arctan(1+x\sqrt{2}) + \arctan(1-x\sqrt{2})\} + C.$$

(m) $\frac{1}{x^6-1}$

$$x^6 - 1 = (x^3-1)(x^3+1) = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$$

$$\frac{1}{x^6-1} = \frac{1}{6}\{\frac{1}{x-1} - \frac{1}{x+1} - \frac{x+2}{x^2+x+1} + \frac{x-2}{x^2-x+1}\}$$

$$I = \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{12}\log\frac{x^2-x+1}{x^2+x+1} - \frac{2}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}(x+\frac{1}{2})$$

$$- \frac{2}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}(x-\frac{1}{2}) + C$$

As a special challenge you may wish to ask for $I = \int \frac{dx}{1+x^6}$ which Leibniz failed to represent in elementary terms. For this, note that

$$1 + x^6 = (1+x^2)(1-x^2+x^4)$$

$$= (1+x^2)(1-x\sqrt{3}+x^2)(1+x\sqrt{3}+x^2).$$

In this case,

$$I = \frac{\arctan x}{3} + \frac{1}{4\sqrt{3}}\log\frac{x^2+x\sqrt{3}+1}{x^2-x\sqrt{3}+1} + \frac{1}{6}\arctan(2x+\sqrt{3})$$

$$+ \frac{1}{6}\arctan(2x-\sqrt{3}) + C.$$

2. Prove from Equation (3) that if $Q(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$, where $a_1 < a_2 < \ldots < a_n$, then $\frac{1}{Q(x)}$ has a decomposition into partial fractions of the form

$$\frac{1}{Q(x)} = \frac{r_1}{x - a_1} + \frac{r_2}{x - a_2} + \ldots + \frac{r_n}{x - a_n}.$$

From Equation (3),

$$\frac{1}{Q(x)} = \frac{1}{a_1 - a_2}\left(\frac{1}{x - a_1} - \frac{1}{x - a_2}\right)\frac{1}{(x - a_3) \ldots (x - a_n)}.$$

Thus $\frac{1}{Q(x)} = \left(\frac{1}{P_1(x)} - \frac{1}{P_2(x)}\right)\frac{1}{a_1 - a_2}$ where $P_1$ and $P_2$ are each products of $n - 1$ linear factors. Repeat the process, reducing the numbers of factors in each denominator by 1 at each step. After $n - 1$ steps, collect terms.

3. Prove if

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_0$$

for all but finitely many numbers $x$, that the coefficients of like powers on the right and left are equal; i.e., $a_k = b_k$ for $k = 0, 1, \ldots, n$.

Consider $P(x) = \sum_{k=0}^{n} (a_k - b_k)x^k$. Since $P(x) = 0$ for more than $n$ numbers $x$ it follows since a polynomial of degree $n$ can have at most $n$ roots, that all the coefficients $a_k - b_k$ of $P$ must vanish; otherwise a polynomial of degree less than or equal to $n$ would have more than $n$ roots.

4. Verify that $\int \frac{px + q}{[(x + a)^2 + b^2]} dx$, $b > 0$, can be expressed as the sum of terms of the forms (11a, b, c).

As in the text, substitute $x = a + b \tan \theta$.

$$I = \frac{pa + q}{b} \arctan \frac{x - a}{b} + \frac{p}{2} \log [(x - a)^2 + b^2] + C.$$

## A4-4. Definite Integrals

### Solutions Exercises A4-4a

1. $\int_{-99}^{99} \frac{\sin^{99} \frac{x}{99}}{x^2 + 99^2} dx.$

$I = 0$; the integrand is odd.

2. $\int_0^1 x^3 e^{-3x^2} dx$

Substitute $u = 3x^2$.

$$I = \frac{1}{18}\int_0^3 ue^{-u}\,du = -\frac{1}{18}(u+1)e^{-u}\Big|_0^3 = \frac{1}{18}(1 - \frac{4}{e^3}).$$

3. $\int_1^e \log^3 x\,dx$

Integrate by parts with $dv = dx$, $u = \log^k x$, repeatedly, to obtain

$$I = 2(3 - e).$$

4. $\int_0^{\pi/2} \sin^m x\,dx$

From Exercises A4-2, Number 4(a),

$$I_m = \frac{m-1}{m} I_{m-2}.$$

Since $I_0 = \frac{\pi}{2}$ and $I_1 = 1$, if $m$ is even, $I = \frac{\pi}{2}\frac{1 \cdot 3 \cdot 5 \cdots m-1}{2 \cdot 4 \cdot 6 \cdots m}$; if $m$ is odd, $I = \frac{2 \cdot 4 \cdot 6 \cdots m-1}{3 \cdot 5 \cdot 7 \cdots m}$.

5. $\int_0^{\pi/2} \sin^m x \cos^m x\,dx$, ($m$, a positive integer).

$$I = \int_0^{\pi/2} \frac{\sin^m 2x}{2^m} dx$$

$$= \frac{1}{2^{m+1}}\int_0^{\pi} \sin^m \theta\,d\theta$$

where $\theta = 2x$. As in Number 4, $I_m = \frac{m-1}{m} I_{m-2}$; but here $I_0 = \pi$ and $I_1 = 2$.

6. $\int_0^{\pi/2} \frac{dx}{a + b \cos x}$, $a > b \geq 0$.

Set $x = 2 \arctan t$; $\cos x = \frac{1 - t^2}{1 + t^2}$, $dx = \frac{2\,dt}{1 + t^2}$.

$$I = \int_0^1 \frac{2}{(a + b) + (a - b)t^2}\,dt = \frac{2}{\sqrt{a^2 - b^2}} \arctan \sqrt{\frac{a - b}{a + b}}.$$

7. $\int_0^{\pi/2} \sin^7 x \cos^3 x\,dx$

$$I = \int_0^{\pi/2} \sin^7 x\,(1 - \sin^2 x) \cos x\,dx = \frac{1}{8} - \frac{1}{10} = \frac{1}{40}.$$

8. $\int_1^2 \frac{dx}{x + x^5}$

$$I = \int_1^2 \frac{x^{-5}}{1 + x^{-4}}\,dx$$

Set $t = x^{-4}$. Then

$$I = \frac{1}{5} \int_{1/16}^1 \frac{dt}{1 + t} = \frac{1}{5} \log \frac{32}{17}.$$

9. $\int_0^b \sqrt{b^2 - x^2}\,dx = b^2 \int_0^{\pi/2} \cos^2 t\,dt = \frac{\pi}{4} b^2$ where $t = b \sin x$.

10. $\int_{-\pi/4}^{\pi/4} \frac{\sin^5 \theta + 1}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}\,d\theta$, $a > 0$, $b > 0$.

$$I = \int_{-\pi/4}^{\pi/4} \frac{1}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}\,d\theta$$

since $\frac{\sin^5 \theta}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$ is odd. Set $t = \tan \theta$ to obtain

$$I = \int_{-1}^1 \frac{dt}{b^2 + a^2 t^2} = \frac{2}{ab} \arctan \frac{a}{b}.$$

11. Compare $\int_0^a f(x)dx$ with $\int_{-a}^0 f(x)dx$ when $f$ is even or odd to derive the results (1) and (2) of the text by a method other than th[illegible] you employed for Exercises A4-2, Number 4.

Substitute $x = -t$ in $\int_0^a f(x)dx$ to obtain

$$\int_0^a f(x)dx = \int_{-a}^0 f(-x)dx.$$

Hence,

$$\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$$

$$= \begin{cases} 0, & \text{if } f \text{ is odd,} \\ 2\int_0^a f(x)dx, & \text{if } f \text{ is even.} \end{cases}$$

12. Prove if $f$ is integrable and periodic of period $p$, then for all $a$ and $b$

$$\int_a^{a+p} f(x)dx = \int_b^{b+p} f(x)dx.$$

Follow the geometrical approach of the text. Set $k = 1 + \left[\!\left[\frac{b-a}{p}\right]\!\right]$. Then $b < a + kp \le b + p$. Consequently,

$$I = \int_b^{b+p} f(x)dx = \int_b^{a+kp} f(x)dx + \int_{a+kp}^{b+p} f(x)dx.$$

~~Now in the integral from b to a + kp~~, make the substitution $u = x - (k - 1)p$; in the integral from $a + kp$ to $b + p$, $u = x - kp$. Then

$$I = \int_{b-(k-1)p}^{a+p} f(u + kp)du + \int_a^{b-(k-1)p} f(u + (k - 1)p)du$$

$$= \int_{b-(k-1)p}^{a+p} f(u)du + \int_a^{b-(k-1)p} f(u)du$$

$$= \int_a^{a+p} f(u)du,$$

by Theorem A4-2b.

13. Prove that if $n \geq 2$ then

$$.500 < \int_0^{1/2} \frac{dt}{\sqrt{1 - t^n}} < .524.$$

If $0 \leq t \leq 1$, then $0 \leq t^n \leq t^2 \leq 1$ and $0 \leq 1 - t^2 \leq 1 - t^n \leq 1$. Thus,

$$.500 = \int_0^{1/2} dt < \int_0^{1/2} \frac{dt}{\sqrt{1 - t^n}} \leq \int_0^{1/2} \frac{dt}{\sqrt{1 - t^2}}$$

$$\leq \arcsin \frac{1}{2} \leq \frac{\pi}{6} < .524.$$

14. Prove that $\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx = \pi^2$.

Since $\frac{x}{1 + \cos^2 x}$ is odd; $\frac{x \sin x}{1 + \cos^2 x}$, even,

$$I = 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = 4 \int_0^{\pi} \frac{x \sin x}{2 - \sin^2 x} dx$$

$$= 2\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx.$$

Hence,

$$I = -2\pi \arctan \cos x \Big|_0^{\pi} = -2\pi(-\frac{\pi}{4} - \frac{\pi}{4}) = \pi^2.$$

15. Show $\frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots \frac{(2n)^2}{(2n - 1)(2n + 1)} = \frac{1}{2n + 1} \left[\frac{2^{2n}(n!)^2}{(2n)!}\right]^2$.

First, observe that $2 \cdot 4 \cdot 6 \cdots (2n) = 2^n(n!)$; then that

$$1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n - 1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{2n!}{2^n(n!)}.$$

16. Determine the value exact to three decimal places of

$$\int_1^{e^{36.1}} \frac{\sin(\pi \log x)}{x}\,dx.$$

Set $\theta = \pi \log x$.

$$I = \frac{1}{\pi}\int_0^{36.1\pi} \sin\theta\, d\theta = -\frac{1}{\pi}\cos\theta \Big|_0^{\pi/10}$$

where $\int_0^{2\pi} \sin\theta\, d\theta = 0$ is used. Then

$$\cos\frac{\pi}{10} = 1 - \frac{\pi^2}{200} + \epsilon$$

where $0 < \epsilon \le \frac{1}{24}\left(\frac{\pi}{10}\right)^4$ Since $\pi^2 < 10$, the error term may be neglected to the desired accuracy. Hence, to the nearest thousandth,

$$I = \frac{\pi}{200} = .016.$$

17. Evaluate $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t + \frac{\pi}{4}}{2 - \cos 2t}\,dt.$

Observe that $\dfrac{t}{2 - \cos 2t}$ is odd; hence

$$I = \frac{\pi}{4}\int_{-\pi/4}^{\pi/4} \frac{dt}{2 - \cos 2t}.$$

Set $u = \tan t$, $\cos 2t = \dfrac{1 - \tan^2 t}{1 + \tan^2 t} = \dfrac{1 - u^2}{1 + u^2}$, $dt = \dfrac{du}{1 + u^2}$. Then

$$I = \frac{\pi}{2}\int_0^1 \frac{du}{1 + 3u^2} = \frac{\pi}{2\sqrt{3}} \arctan u\sqrt{3}\,\Big|_0^1 = \frac{\pi^2}{6\sqrt{3}}.$$

## Appendix 5

## AREA AND INTEGRAL

### Solutions Exercises A5-1

1. Prove from Property 3 that if a region $R$ is the union of $n$ non-overlapping regions then

$$\alpha(R) = \alpha(R_1) + \alpha(R_2) + \dots + \alpha(R_n).$$

We have

$$\begin{aligned}\alpha(R_1 \cup R_2 \cup R_3 \cup \dots \cup R_n) &= \alpha(R_1) + \alpha(R_2 \cup R_3 \cup \dots \cup R_n)\\ &= \alpha(R_1) + \alpha(R_2) + \alpha(R_3 \cup \dots \cup R_n)\\ &= \dots\\ &= \alpha(R_1) + \alpha(R_2) + \alpha(R_3) + \dots + \alpha(R_n).\end{aligned}$$

The argument may be formalized by the use of mathematical induction.

2. Show that Property 2 is actually a consequence of Property 3 given that area is nonnegative. Incorporate the notion of complementary regions.

Let $S$ be a subregion of $T$ and let $R$ be the complementary region of $S$ in $T$; i.e., $R$ is the region which does not overlap $S$ and for which $R \cup S = T$. (We deliberately omit the question of existence of $R$.) Since $\alpha(R) \geq 0$ and

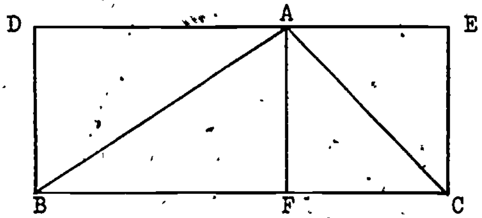
$$\alpha(R) + \alpha(S) = \alpha(T)$$

we have the desired result

$$\alpha(S) = \alpha(T) - \alpha(R) \leq \alpha(T).$$

3. (a) Using the given properties of area obtain the area of a triangle by elementary geometrical arguments.

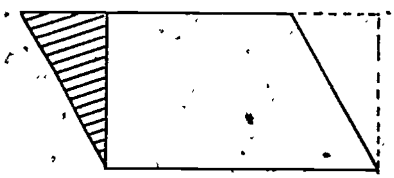
Let $\Delta ABC$ be the triangle and let $BC$ be the longest side. We inscribe $\Delta ABC$ in a rectangle with one side on $\overline{BC}$ and suppose the foot of the perpendicular from $A$ to $BC$ lies on $BC$ at $F$. (See Figure.)




From elementary geometry we have $\Delta AFC$ congruent to $\Delta CEA$, and $\Delta AFB$ to $\Delta BDA$. It follows that the area of the triangle $ABC$ is half the area of the rectangle $BCED$ and hence equal to half the product of the base $BC$ and the altitude $AF$.

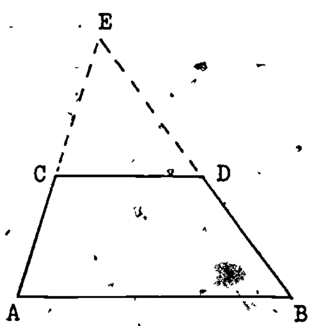
In this "proof" it is assumed from elementary geometry that congruent regions have equal area.

(b) Do the same for a trapezoid.



(i) If the sides of the trapezoid are parallel, the trapezoid is a parallelogram and has the same area as a rectangle of the same base and altitude.

(ii) If the sides are not parallel, extend them until they meet. The area is then the difference between the area of $\Delta EAB$ and $\Delta ECD$. In either case we get the usual formula.



4. If Property 4 is replaced by

Property $4^*$ : The area of a unit square is one,
Property 5 : Congruent regions have the same area,

show that the area of a square whose side is of length $a$ is $a^2$.

The proof is given first for rational $a$ then for arbitrary real $a$. If $a = \frac{1}{n}$ for a natural number $n$, then from the observation that the unit square can be subdivided into $n^2$ squares of sidelength $\frac{1}{n}$ it follows from Property 3 and new Properties $4^*$ and 5 that the area of the square is $\frac{1}{n^2}$. If $a = \frac{m}{n}$, then the square may be subdivided into $m^2$ congruent squares of sidelength $\frac{1}{n}$ and from the preceding result, the area is $\frac{m^2}{n^2}$. If now $a$ is any real number, take $m = [\![an]\!]$. Then $m \leq an < m + 1$. It follows that the given square contains a square of sidelength $\frac{m}{n}$ and is contained in a square of sidelength $\frac{m+1}{n}$. Consequently, for the area $A$ of the given square, by Property 2,

$$\frac{m^2}{n^2} \leq A < \frac{(m+1)^2}{n^2}.$$

Consequently, for all natural numbers $n$,

$$\frac{(an-1)^2}{n^2} < A < \frac{(an+1)^2}{n^2},$$

or

$$a^2 - \frac{2a}{n} + \frac{1}{n^2} < A < a^2 + \frac{2a}{n} + \frac{1}{n^2};$$

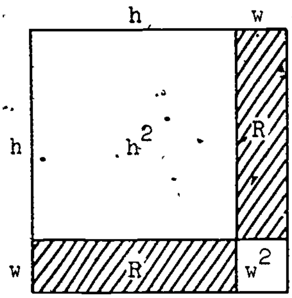
hence

$$|A - a^2| < \frac{2a+1}{n},$$

from which the result follows.

5. Using Number 4, show that the area of a rectangle of height $h$ and width $w$ is $hw$.

Form the square of side $h + w$. If $A$ is the area of the given rectangle, then from the figure and from Number 4 $(h+w)^2 = 2R + h^2 + w^2$. It follows that $R = hw$.



## Solutions Exercises A5-2

1. (Requires Section A3-2(ii) for parts (d) and (e).) Use the summation method to find the area of the standard region defined by
(a) $f : x \to c, \quad 0 \leq x \leq b, \quad c > 0.$

Use a subdivision of the interval into $n$ equal parts. Define $\alpha(S)$, $\alpha(T)$, and $\alpha(R)$ as in the text, using the respective minimum and maximum values of $f$ in each subinterval. In this case the maximum on any interval is equal to the minimum, so that

$$\alpha(S) = \alpha(T) = \sum_{k=1}^{n} c \cdot \frac{b}{n} = bc.$$

(b) $f : x \to cx, \quad 0 \leq x \leq b, \quad c > 0.$

Here

$$\alpha(S) = \sum_{k=1}^{n} c \cdot \frac{(k-1)b}{n} \cdot \frac{b}{n} = \frac{cb^2}{n^2} \sum_{k=1}^{n} (k-1)$$

$$= \frac{cb^2}{n^2} \frac{n(n-1)}{2},$$

and

$$\alpha(T) = \sum_{k=1}^{n} c \cdot \frac{kb}{n} \cdot \frac{b}{n} = \frac{cb^2}{n^2} \sum_{k=1}^{n} k = \frac{cb^2}{n^2} \frac{n(n+1)}{2}.$$

From $\alpha(S) \leq \alpha(R) \leq \alpha(T)$ it follows that

$$-\frac{cb^2}{2n} \leq \alpha(R) - \frac{cb^2}{2} \leq \frac{cb^2}{2n},$$

for each natural number $n$. Consequently,

$$\alpha(R) = \frac{cb^2}{2}.$$

(c) $f : x \to x^2 + 2x, \quad 0 \le x \le b.$

Here

$$\alpha(T) = \sum_{k=1}^{n}\left[\left(\frac{kb}{n}\right)^2 + 2\,\frac{kb}{n}\right]\frac{b}{n}$$

$$= \frac{b^3}{n^3}\sum_{k=1}^{n} k^2 + \frac{2b^2}{n^2}\sum_{k=1}^{n} k$$

$$= \frac{b^3}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) + \frac{2b^2}{n^2}\,\frac{n(n+1)}{2}$$

$$= \frac{b^3}{3} + b^2 + \frac{b^2}{n} + \frac{b^3}{2n} + \frac{b^3}{6n^2}.$$

and

$$\alpha(S) = \alpha(T) - \frac{b^3}{n} - \frac{2b^2}{n}.$$

As in Part (b), from $\alpha(S) \le \alpha(R) \le \alpha(T)$ obtain

$$\alpha(R) = \frac{b^3}{3} + b^2.$$

(d) $f : x \to \sin(ax + b), \quad 0 \le x \le c;$ $a, b, c$ such that $\sin(ax + b) \ge 0$ on $[0,c]$.

The interval $[0,c]$ may be subdivided into two subintervals where $f$ is strongly monotone. For simplicity, assume $f$ is increasing on $[0,c]$. Then

$$\alpha(T) = \sum_{k=1}^{n} \frac{c}{n}\sin\left(\frac{akc}{n} + b\right).$$

In A3-2(ii), Equation (6), take $\frac{ac}{n}$ for $a$ and $b + \frac{ac}{2n} - \frac{\pi}{2}$ for $b$.

Then

$$\alpha(T) = \frac{c}{n} \sum_{k=1}^{n} \sin\left(\frac{akc}{n} + b\right) = \frac{c \sin\left(\frac{ac}{2} + b + \frac{ac}{2n}\right) \sin \frac{ac}{2}}{n \sin \frac{ac}{2n}} .$$

Consequently, from the continuity of $\cos$ and $\lim\limits_{x \to 0} \frac{\sin x}{x}$,

$$\lim_{n \to \infty} \alpha(T) = \frac{2}{a} \sin\left(\frac{ac}{2} + b\right) \sin \frac{ac}{2} = \frac{\cos(ac + b) - \cos b}{a} .$$

A similar argument yields the same limit for $\alpha(S)$. By the Squeeze Theorem, it follows that $\alpha(R)$ is this common limit.

(e) $f : x \to \cos^2 x, \quad 0 \le x \le c.$
Use $\cos^2 x = \frac{\cos 2x + 1}{2}$. Proceed as in Part (d) with $\cos 2x = \sin(2x + \frac{\pi}{2})$.

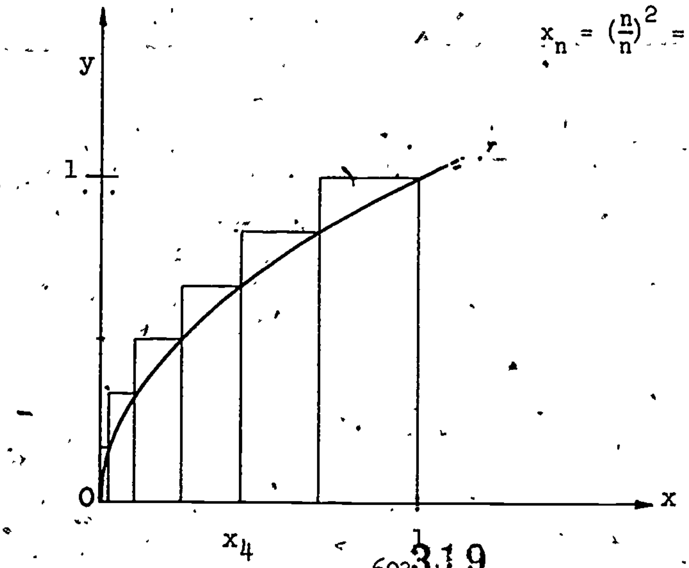
$$\alpha(R) = \frac{\sin 2c}{4} + \frac{c}{2} .$$

2. Determine the area of the standard region $f : x \to \sqrt{x}$ on $[0,1]$. (The summation encountered will be similar to the one encountered in this section.)

In order to avoid a sum which involves square roots of natural numbers, it is most convenient to subdivide the interval $[0,1]$ in the following way:

$$x_0 = 0,\ x_1 = \left(\frac{1}{n}\right)^2,\ x_2 = \left(\frac{2}{n}\right)^2,\ \ldots,\ x_{n-1} = \left(\frac{n-1}{n}\right)^2,$$

$$x_n = \left(\frac{n}{n}\right)^2 = 1 .$$



For this subdivision

$$f(x_k) = \sqrt{\left(\frac{k}{n}\right)^2} = \frac{k}{n}.$$

Since $f : x \to \sqrt{x}$ is increasing on $[0,1]$

$$f(x_{k-1}) \le f(x) \le f(x_k) \quad \text{on} \quad [x_{k-1}, x_k],$$

and the upper sum has the form

$$\alpha(T) = \sum_{k=1}^{n} f(x_k)[x_k - x_{k-1}]$$

$$= \sum_{k=1}^{n} \frac{k}{n}\left[\left(\frac{k}{n}\right)^2 - \left(\frac{k-1}{n}\right)^2\right]$$

$$= \frac{1}{n^3}\sum_{k=1}^{n} k(2k - 1)$$

$$= \frac{1}{n^3}\left[2\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k\right]$$

$$= \frac{1}{n^3}\left[\frac{2n^3}{3} + n^2 + \frac{n}{3} - \frac{n(n+1)}{2}\right]$$

$$= \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}.$$

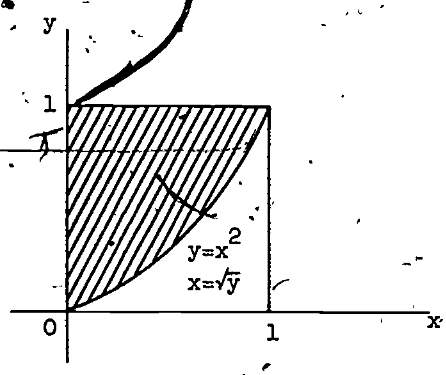
We have also

$$\alpha(S) = \alpha(T) - \frac{1}{n} = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}.$$

It follows that the area is $\frac{2}{3}$.

3. Obtain the result of Exercise 2 using only the fact that the area under $f : x \to x^2$ on $[0,1]$ is $\frac{1}{3}$, together with the basic properties of area, without resort to summation techniques.

By Property 3, the area of the standard region under the graph of $x = \sqrt{y}$ on $[0,1]$ (the shaded region) plus the area of the standard region under the graph $y = x^2$ on $[0,1]$ (unshaded region) is 1.

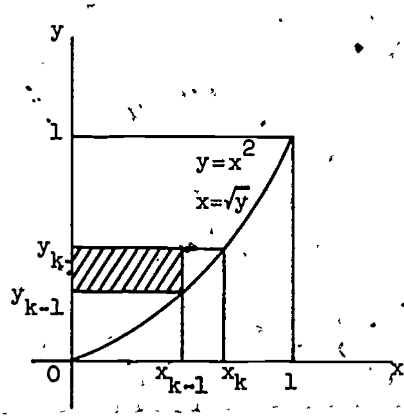



4. Show how the upper estimating sums for $\sqrt{x}$ are related term-by-term to the lower estimating sums for $x^2$. (Hint: Sketch a graph of $y = x^2$. Use this graph and the y-axis to represent the standard region defined by $\sqrt{x}$.)

Observe from the figure that a term from the upper sum (area of the unshaded rectangle) for $f : x \to x^2$ of the form $x_k^2(x_k - x_{k-1})$, corresponds to a term of the lower sum (area of the shaded rectangle) for $g : y \to \sqrt{y}$ of the form

$$\sqrt{y_{k-1}}\,(y_k - y_{k-1})$$

where $y_k = x_k^2$. Furthermore, the sum of the two is $x_k y_k - x_{k-1} y_{k-1}$. Adding the upper sum for $f$ to the lower sum for $g$, we have




$$(x_1y_1 - x_0y_0) + (x_2y_2 - x_1y_1) + \ldots + (x_ny_n - x_{n-1}y_{n-1})$$
$$= x_ny_n - x_0y_0$$
$$= 1.$$

5. If $S_n = \sqrt{1} + \sqrt{2} + \ldots + \sqrt{n}$, show that

$$\frac{2}{3}\sqrt{n^3} < S_n < \frac{2}{3}\sqrt{n^3} + \sqrt{n} .$$

Divide [0,1] into n equal subintervals. An upper bound for the area $\alpha(R)$ of the standard region $f : x \to \sqrt{x}$ on [0,1] is

$$\alpha(T) = \sum_{k=1}^{n} \sqrt{\frac{k}{n}} \cdot \frac{1}{n} = \frac{1}{\sqrt{n^3}} S_n;$$

a lower bound is

$$\alpha(S) = \sum_{k=1}^{n} \sqrt{\frac{k-1}{n}} \cdot \frac{1}{n} = \frac{1}{\sqrt{n^3}} (S_n - \sqrt{n}).$$

Since

$$\alpha(S) \le \alpha(R) \le \alpha(T)$$

by Property 2, and $\alpha(R) = \frac{2}{3}$ by Number 2, the result follows immediately.

TC. A5-3. Integration by Summation Techniques

Once the student learns the Fundamental Theorem he may come to believe that the original conception of integral as the limit of a sum is not useful for analysis or computation. In this section it is shown that the formal integrals of polynomials and of the circular functions sin and cos can be obtained directly from the definition by summation techniques. This is something of a tour-de-force, but many students find the approach illuminating. Of course, summation remains valuable as a method of getting numerical estimates.

Solutions Exercises A5-3

1. Show simply, without repeating the argument of the text, that the lower sum over $\sigma$, $L = \sum_{k=1}^{n} x_{k-1}^{r}(x_{k-1} - x_k)$, also has the limit (7).

Since $L = U - h^{r+1} n^r = U - h(b-a)^r$, $\lim_{h \to 0} L = \lim_{h \to 0} U = \frac{a^{r+1}}{r+1}$.

2. Employ Equation (8) of Section A3-2(ii) to obtain $\int_0^a \sin x \, dx$ for $0 < a \leq \frac{\pi}{2}$.

Replace $a$ by $h = \frac{a}{n}$ in Equation (8) of Section A3-2(ii), and note that

$$\sin \frac{(n+1)h}{2} \sin \frac{nh}{2} = \cos \frac{h}{2} - \cos\left(n + \frac{1}{2}\right)h$$

to obtain

$$\sum_{k=1}^{n} \sin kh = \frac{\cos \frac{h}{2} - \cos\left(n + \frac{1}{2}\right)h}{2 \sin \frac{h}{2}} .$$

Since $\sin x$ is increasing on $[0,a]$ the upper and lower sums for a subdivision of the interval into $n$ equal parts are given by

$$U = \sum_{k=1}^{n} h \sin kh \quad \text{and} \quad L = \sum_{k=1}^{n} h \sin (k - 1)h.$$

Note that $L = U - h \sin a$. Insert $n = ah$ in the formula for $\sum_{k=1}^{n} \sin kh$ to obtain for the upper sum,

$$U = \frac{\frac{1}{2} h}{\sin \frac{1}{2} h} [\cos \tfrac{1}{2} h - \cos (a + \tfrac{1}{2} h)].$$

Use the continuity of the cosine and

$$\lim_{h \to 0} \frac{\frac{1}{2} h}{\sin \frac{1}{2} h} = 1$$

to obtain

$$\lim_{h \to 0} U = 1 - \cos a = \lim_{h \to 0} L;$$

whence

$$\int_0^a \sin x \, dx = 1 - \cos a.$$

Solutions Exercises A5-4

1. By using upper and lower sum estimates evaluate the integral of each function $f$ over the indicated interval.

(a) $f(x) = 2 - x^2$ $\qquad 0 \le x \le 1$

(b) $f(x) = x$ $\qquad 1 \le x \le 2.5$

(c) $f(x) = \frac{5}{2}$ $\qquad 2.5 \le x \le 3$

(d) $f(x) = 5 - x$ $\qquad 3 \le x \le 5$

In each of the following we use subdivision into $n$ equal parts.

(a) Since $f$ is monotone decreasing, we take

$$L = \sum_{k=1}^{n} (2 - x_k^2)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} \left(2 - \frac{k^2}{n^2}\right)\frac{1}{n} \qquad \left(x_k = \frac{k}{n}\right)$$

$$= \frac{1}{n}\sum_{k=1}^{n} 2 - \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= 2 - \frac{1}{3} - \frac{1}{2n} - \frac{1}{6n^2}.$$

Since $U = L + \frac{1}{n} - \frac{5}{3} + \frac{1}{2n} - \frac{1}{6n^2}$, the integral is $\frac{5}{3}$.

(b) Take $\qquad U = \sum_{k=1}^{n} (1 + kh)h, \qquad (x_k = 1 + kh)$

$$L = U(\sigma) - 1.5h$$

where $\qquad h = \frac{(2.5 - 1)}{n} = \frac{1.5}{n}$

$$U = \sum_{k=1}^{n} h + h^2 \sum_{k=1}^{n} k$$

$$= nh + h^2 \frac{n(n + 1)}{2}.$$

Then, since $nh = 1.5$,

$$U = 1.5 + \frac{(1.5)^2(1 + \frac{1}{n})}{2}.$$

It follows that the integral is

$$1.5 + \frac{2.25}{2} = 2.625.$$

(c) We take for both upper and lower sums

$$U = L = \sum_{k=1}^{n} \frac{5}{2} h$$

where $h = \frac{(3 - 2.5)}{n} = \frac{0.5}{n}$.

$$\sum_{k=1}^{n} \frac{5}{2} h = \frac{5nh}{2} = \frac{5}{2}(0.5) = 1.25.$$

(d) We take for the lower sum

$$L = \sum_{k=1}^{n} [5 - (3 + kh)]h$$

and take $U = L + 2h$ where $h = \frac{(5 - 3)}{n} = \frac{2}{n}$.

$$L = h \sum_{k=1}^{n} (2 - kh)$$

$$= h \sum_{k=1}^{n} 2 - h^2 \sum_{k=1}^{n} k$$

$$= 2nh - \frac{h^2 n(n + 1)}{2}$$

$$= 4 - \frac{4(1 + \frac{1}{n})}{2}$$

$$= 2 - \frac{2}{n}$$

It follows that the integral is 2.

2. (a) Find the minimum and the maximum values of $f(x) = 2 + 2x - x^2$ on the interval $[0,1]$, and use them to find two numbers respectively below and above the value of $\int_0^1 f(x)\,dx$.

$f'(x) = 2 - 2x$ is zero when $x = 1$.
$f(0) = 2$ and $f(1) = 3$.
Max: $f(x) = 3$, Min: $f(x) = 2$ for $x$ in $[0,1]$.
Hence $U = 3 \cdot 1 = 3$ and $L = 2 \cdot 1 = 2$.

(b) Check your result by evaluating the integral.

Use the summations of Exercises A5-2, Number 1 and observe that upper and lower sums for $x^2$ are the negatives of lower and upper sums, respectively, for $-x^2$.

$$\int_0^1 f(x)dx = 2 + 1 - \frac{1}{3} = 3 - \frac{1}{3},$$

a value between 2 and 3.

3. Find upper and lower sums differing by less than .1, for the area under $f(x) = \frac{1}{x}$ on the interval $[1,2]$.

Take a subdivision of $[1,2]$ into $n$ equal subintervals and use the maximum and minimum of $\frac{1}{x}$ as bounds in the subinterval, we have

$$U - L = (1 - \frac{1}{2})\frac{1}{n} = \frac{1}{2n}.$$

It is sufficient to take $\frac{1}{2n} < \frac{1}{10}$ or $n > 5$.
Taking $n = 6$ we obtain.

$$U = (1 + \frac{6}{7} + \frac{6}{8} + \frac{6}{9} + \frac{6}{10} + \frac{6}{11}) \cdot \frac{1}{6} = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots + \frac{1}{11},$$

$$L = (\frac{6}{7} + \frac{6}{8} + \frac{6}{9} + \frac{6}{10} + \frac{6}{11} + \frac{6}{12}) \cdot \frac{1}{6} = \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots + \frac{1}{12}.$$

(A decimal representation of the answer is not required.)

4. Evaluate each of the following integrals, using upper and lower sum estimates.

(a) $\int_{-1}^{1} x^3\, dx$

(b) $\int_{-2}^{2} |x|\, dx$

(c) $\int_{-1}^{1} x^2\, dx$

In each of the following use a subdivision into $2n$ equal parts, and set $h = \frac{(b - a)}{2n}$ where $a$ and $b$ are the lower and upper ends of integration respectively, and separate the sums for positive and negative values of $x$.

(a) Separate the upper sums into two sums over the intervals $[-1,0]$ and $[0,1]$.

$$U = \sum_{k=1}^{n} (kh)^3 h - \sum_{i=1}^{n} [(i - 1)^3 h^3] h$$

$$= \sum_{k=1}^{n} (kh)^3 h - \sum_{k=0}^{n-1} (kh)^3 h$$

$$= (nh)^3 h$$

$$= h \qquad \left(\text{from } h = \frac{2}{2n} = \frac{1}{n}\right).$$

Then

$$L = U - 2h = -h.$$

The integral is zero.

(b) Separate the upper sums into sums over the intervals $[-2,0]$, $[0,2]$.

$$U = \sum_{k=1}^{n} |kh| h + \sum_{k=1}^{n} |-kh| h$$

$$= 2 \sum_{k=1}^{n} (kh) h$$

$$= 2h^2 \sum_{k=1}^{n} k$$

$$= h^2 n(n + 1) \qquad (\text{where } hn = 2)$$

$$= 4 + 2h$$

and

$$L = U - 2h - 2h = 4 - 2h.$$

The value of the integral is 4.

(c) By the method of the preceding exercise we find

$$U = 2\sum_{k=1}^{n} (kh)^2 h = 2h^3 \sum_{k=1}^{n} k^2$$

and since $h = \frac{1}{n}$, from Section 6-2, we have

$$U = \frac{2}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \frac{2}{3} + \frac{1}{n} + \frac{1}{3n^2}.$$

$$L = U - 2h = U - \frac{2}{n}.$$

The integral is $\frac{2}{3}$.

5. Approximate $\int_0^1 \frac{1}{1 + x^2}\,dx$ by Riemann sums.

Given a subdivision $\sigma = \{x_0, x_1, \ldots, x_n\}$ of $[0,1]$ we have

$$R = \sum_{k=1}^{n} \frac{1}{1 + \xi_k^2}(x_k - x_{k-1})$$

where $x_{k-1} \le \xi_k \le x_k$. If we choose an equal subdivision and take $\xi_k = x_k$ we obtain the following approximations:

$$n = 1 \quad \frac{1}{1 + 1}\cdot 1 = \frac{1}{2} = 0.50$$

$$n = 2 \quad \left(\frac{1}{1 + \frac{1}{4}} + \frac{1}{1 + 1}\right)\frac{1}{2} = 0.65$$

$$n = 3 \quad \left(\frac{1}{1 + \frac{1}{9}} + \frac{1}{1 + \frac{4}{9}} + \frac{1}{1 + 1}\right)\frac{1}{3} \approx 0.70,$$

(Exact value is $\frac{\pi}{4} = 0.785 \ldots$).

6. A function $f$ defined on the interval $[a,b]$ is said to be a <u>step-function</u> on $[a,b]$ if for some partition $\sigma = \{x_0, x_1, \ldots, x_n\}$ of the interval, $f(x)$ is constant on each open subinterval $(x_{k-1}, x_k)$, $k = 1, 2, \ldots, n$. Thus $\operatorname{sgn} x$ is a step function on $[-1,1]$, where $\operatorname{sgn} x$ is defined by

$$\operatorname{sgn} x = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Find $\int_a^b \operatorname{sgn} x \, dx$.

If $[a,b]$ does not include the origin there are two cases.

(i) $a > 0$. Since $\operatorname{sgn} x = 1$ on $[a,b]$, the integral has the value $b - a$.

(ii) $b < 0$. Since $\operatorname{sgn} x = -1$ on $[a,b]$ the integral has the value $-(b - a)$.

(iii) If $[a,b]$ includes the origin, $a \le 0 \le b$. We take a subdivision and upper and lower sums and obtain the integral $I$ for each case as follows.

$a = 0$, $a = \{0, \epsilon, b\}$,

$$U = b, \quad L = b - \epsilon$$

$$I = b$$

$a < 0 < b$, $\sigma = \{a, -\epsilon, \epsilon, b\}$, where $0 < \epsilon < \min\{-a, b\}$,

$$U = b + a + 2\epsilon$$

$$L = b + a - 2\epsilon$$

$$I = b + a$$

$b = 0$, $\sigma = \{a, -\epsilon, 0\}$

$$U = a + \epsilon, \quad L = a$$

$$I = a$$

These results can be summarized by the simple formula

$$\int_a^b \operatorname{sgn} x \, dx = |b| - |a|.$$

7. Evaluate each of the following integrals: The function $[x]$ is defined in Appendix 1.

(a) $\int_{-1}^{3} [\![3x + 4]\!]\,dx$ (c) $\int_{1}^{5} \sqrt{2[\![x]\!]}\,dx$

(b) $\int_{0}^{10} [\![\frac{x}{4}]\!]\,dx$ (d) $\int_{1}^{5} [\![\sqrt{2x}]\!]\,dx$

Each of the given integrands is an increasing step-function and hence is integrable either by the monotone property or by Number 6. In the notation of the solution of Number 6a, the integral of a step-function is

$$\sum_{k=1}^{n} c_k(x_k - x_{k-1})$$

as can be deduced directly from the given upper and lower estimates. Apply this result as follows.

(a) $\int_{-1}^{3} [\![3x + 4]\!]\,dx = (1 + 2 + 3 + \ldots + 12) \cdot \frac{1}{3} = 26$

(b) $\int_{0}^{10} [\![\frac{x}{4}]\!]\,dx = (0 + 1) \cdot 4 + 2 \cdot 2 = 8$

(c) $\int_{1}^{5} \sqrt{2[\![x]\!]}\,dx = (\sqrt{2} + \sqrt{4} + \sqrt{6} + \sqrt{8}) \cdot 1$

$$= 2 + 3\sqrt{2} + \sqrt{6}$$

(d) $\int_{1}^{5} [\![\sqrt{2x}]\!]\,dx = 1 \cdot 1 + 2 \cdot \frac{5}{2} + 3 \cdot \frac{1}{2} = \frac{15}{2}$

8. Show that $\int_{a}^{a} f(x)dx = 0$.

The integral can be calculated by subdividing the interval $[a,a]$, and calculating the appropriate Riemann sums. But all subintervals of $[a,a]$ are of length zero, and any element of the Riemann sum,

$$f(x_k)[a - a] = 0.$$

Alternately, the Fundamental Theorem of Calculus states that

$$\int_{a}^{a} f(x)dx = F(a) - F(a) = 0 ,$$

where $F'(x) = f(x)$.

1. Exhibit the details of the proof of Part (i) when $\alpha < 0$.

If $m_k \leq f(x) \leq M_k$ on $[x_{k-1}, x_k]$ and $\alpha < 0$, then multiply by $\alpha$ to obtain

$$\alpha M_k \leq \alpha f(x) \leq \alpha m_k.$$

From $\alpha \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \leq \alpha \sum_{k=1}^{n} m_k(x_k - x_{k-1})$ obtain the lower sum $\alpha L$ and upper sum $\alpha U$ for $f$ over $[a,b]$. If $U - L < \epsilon$, then

$$0 < \alpha L - \alpha U < |\alpha|\epsilon.$$

Observe that $\alpha f$ is then integrable by Theorem A5-4a. If $I$ is the integral of $f$ and $J$ that of $\alpha f$ over $[a,b]$, then

$$0 \leq U - I < \epsilon, \quad 0 \leq J - \alpha U < |\alpha|\epsilon,$$

from which it follows that

$$\begin{aligned} |J - \alpha I| &= |(J - \alpha U) + \alpha(U - I)| \\ &\leq |J - \alpha U| + |\alpha| \cdot |U - I| \\ &< 2|\alpha|\epsilon. \end{aligned}$$

This result holds for all positive $\epsilon$, hence $J = \alpha I$.

2. (a) If the graph of $f$ is symmetric with respect to the origin, then $f$ is odd. Prove that if $f$ is odd and integrable on $[-a,a]$, then

$$\int_{-a}^{a} f(x)dx = 0.$$

(b) If the graph of $f$ is symmetric with respect to the y-axis, then $f$ is even. Prove for an even function $f$ which is integrable on $[-a,a]$ that

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

Interpret this result geometrically.

Take a subdivision of the interval into $2n$ equal parts and set $h = \frac{a}{n}$. Take the Riemann sum

$$R = \sum_{k=1}^{n} f(-\xi_k)h + \sum_{k=1}^{n} f(\xi_k)h.$$

where $(k - 1)h \le \xi_k \le kh$, $(k = 1, \ldots, n)$.

(a) If $f$ is odd, $f(-x) = -f(x)$, and

$$R = \sum_{k=1}^{n} [-f(\xi_k)h] + \sum_{k=1}^{n} f(\xi_k)h$$

$$= 0.$$

Since the limit of the Riemann sums is the same independently of the method of subdivision and the choice of $\xi_k$ it follows that the integral is 0.

(b) If $f$ is even, $f(-x) = f(x)$, and $R = 2\sum_{k=1}^{n} f(\xi_k)h$ where the sum is the Riemann sum for $f$ over the half interval $[0,a]$, and the result follows on taking the limit. Geometrical interpretation: The area of a standard region is equal to that of its mirror image. Parts (a) and (b) can also be done by comparing upper and lower sums on the half-intervals.

3. Prove Theorem A5-5c as a consequence of Lemmas A5-5a and A5-5b. Conversely, derive the Lemmas as corollaries of Theorem A5-5c.

Proof of Theorem A5-5c.
Let $f$ and $g$ be integrable over $[a,b]$. From Lemmas A5-5a and A5-5b applied in succession

$$\alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx = \int_a^b \alpha f(x)\,dx + \int_a^b \beta\, g(x)\,dx$$

$$= \int_a^b [\alpha\, f(x) + \beta\, g(x)]\,dx.$$

Proof of Lemma A5-5a. Take $\beta = 0$ in Theorem A5-5a.
Proof of Lemma A5-5b. Take $\alpha = \beta = 1$ in Theorem A5-5a.

4. Prove: If $f$ and $g$ are integrable where $g : x \to |f(x)|$ on $[a,b]$, then

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx.$$

For all $x$ in $[a,b]$

$$-|f(x)| \le f(x) \le |f(x)|,$$

whence by Theorem A5-5a

$$\int_a^b -|f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

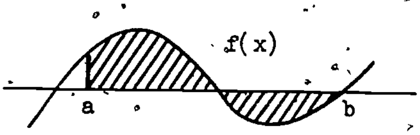
or

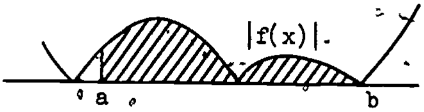
$$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

or

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx.$$

We observe that $\int_a^b |f(x)|dx$ represents the sum of the areas of the regions bounded by the graph of $f$, the x-axis, and the vertical lines, $x = a$, $x = b$.



5. Compute the values of the given integrals using Theorem A5-5c.

(a) $\int_2^3 (3x^2 - 5x + 1)dx$

$$\int_2^3 (3x^2 - 5x + 1)dx = 3\int_2^3 x^2\, dx - 5\int_2^3 x\, dx + \int_2^3 dx.$$

From Examples A5-5a, b, c, and

$$\int_a^b f(x)dx = \int_0^b f(x)dx - \int_0^a f(x)dx$$

it follows that the integral is

$$3\left(\frac{3^3}{3} - \frac{2^3}{3}\right) - 5\left(\frac{3^2}{2} - \frac{2^2}{2}\right) + (3 - 2) = 7\frac{1}{2}.$$

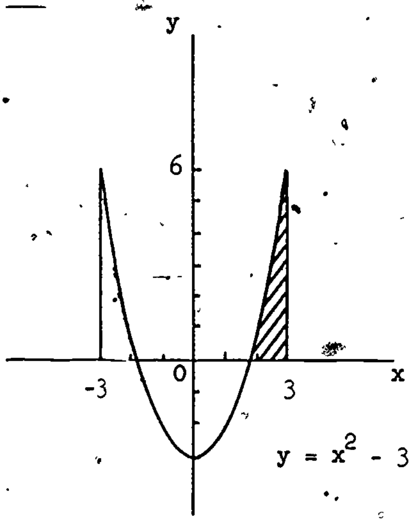
(b) $\int_0^2 (x - 1)(x + 2)dx$

$$\begin{aligned}\int_0^2 (x - 1)(x + 2)dx &= \int_0^2 (x^2 + x - 2)dx \\ &= \frac{2^3}{3} + \frac{2^2}{2} - 2(2) \\ &= \frac{2}{3}.\end{aligned}$$

(c) $\int_{-2}^3 (x + 2)(x - 3)dx$

$$\begin{aligned}\int_{-2}^3 (x + 2)(x - 3)dx &= \int_{-2}^3 (x^2 - x - 6)dx \\ &= \left(\frac{3^3}{3} - \frac{(-2)^3}{3}\right) - \left(\frac{3^2}{2} - \frac{(-2)^2}{2}\right) - 6\left(3 - (-2)\right) \\ &= -\frac{125}{6}.\end{aligned}$$

6. (a) Find the area of the region below the parabola $y = x^2 - 3$ above the x-axis and between the lines $x = -3$, $x = 3$.

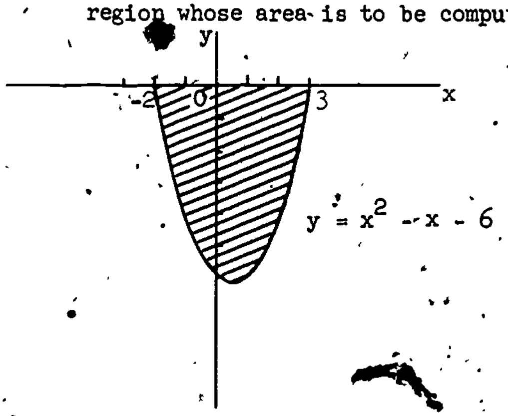



$$\text{Area} = 2\int_{\sqrt{3}}^{3} (x^2 - 3)dx$$

$$= 2\left[\left(\frac{3^3}{3} - \frac{(\sqrt{3})^3}{3}\right) - 3(3 - \sqrt{3})\right]$$

$$= 4\sqrt{3}.$$

(b) Find the area of the region between the graph of $f : x \to x^2 - x - 6$, the x-axis, and the lines $x = -2$, $x = 3$. First draw a rough sketch of $f$ and indicate (by shading) the region whose area is to be computed.




$$\text{Area} = -\int_{-2}^{3} (x^2 - x - 6)dx$$

$$= \frac{125}{6}.$$

(See No. 7c.)

7. Find all values of $a$ for which

$$\int_0^a (x + x^2)dx = 0.$$

The number $a$ must satisfy $\frac{a^3}{3} + \frac{a^2}{2} = 0$. This equation has two solutions

$$a = -\frac{3}{2} \text{ and } a = 0.$$

8. Compute $\int_0^3 f(x)dx$ where

$$f(x) = \begin{cases} 2 - x^2, & 0 \leq x \leq 1 \\ 5 - 4x, & 1 \leq x \leq 3 \end{cases}$$

$$\int_0^3 f(x)dx = \int_0^1 (2 - x^2)dx + \int_1^3 (5 - 4x)dx$$

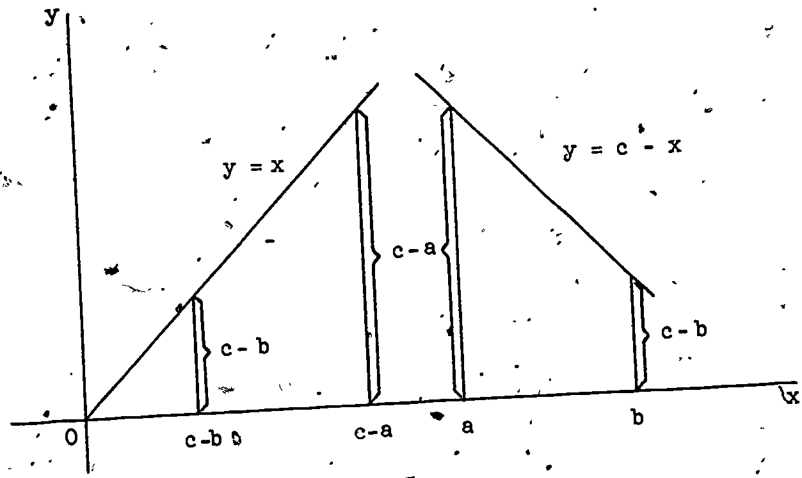
$$= (2 - \tfrac{1}{3}) + [10 - 4(\tfrac{3^2}{2} - \tfrac{1}{2})]$$

$$= \frac{-13}{3}.$$

9. Verify that the following property holds for $f : x \to x$

$$\int_a^b f(c - x)dx = \int_{c-b}^{c-a} f(x)dx.$$

Explain the property geometrically in terms of areas. Do you think that the property holds for other functions that are integrable? Justify your answer.




The integrals represent areas of mirror-image regions. The property is general for a function $f$ integrable over $[c - b, c - a]$. For $f(x) = x$, particularly,

$$\int_a^b (c - x)dx = c(b - a) - \tfrac{1}{2}(b^2 - a^2),$$

$$\int_{c-b}^{c-a} x\,dx = \frac{1}{2}[(c - a)^2 - (c - b)^2]$$

$$= \frac{1}{2}[2bc - 2ac + a^2 - b^2]$$

$$= c(b - a) - \tfrac{1}{2}(b^2 - a^2).$$

10. If a function $f$ is periodic with period $\lambda$ and integrable for all $x$, show that

$$\int_a^{a+n\lambda} f(x)dx = n\int_a^{a+\lambda} f(x)dx$$

($n$, integer). Interpret geometrically.

$$\int_a^{a+n\lambda} f(x)dx = \sum_{k=1}^{n} \int_{a+(k-1)\lambda}^{a+k\lambda} f(x)dx.$$

Now consider the subdivision of the interval $[a + (k - 1)\lambda, a + k\lambda]$ into $m$ equal parts by means of the partition $\{u_0, u_1, \ldots, u_m\}$ where $u_i = a + (k - 1)\lambda + \frac{i\lambda}{m}$ and form the Riemann sum

$$R_k = \sum_{i=1}^{m} f(u_i)\frac{\lambda}{m}$$

$$= \sum_{i=1}^{n} f\left(a + (k - 1)\lambda + \frac{i\lambda}{m}\right)\frac{\lambda}{m}$$

$$= \sum_{i=1}^{n} f(a + \frac{i\lambda}{m})\frac{\lambda}{m}.$$

Since the Riemann sums $R_k$, $(k = 1, \ldots, n)$ for each of the integrals are the same, it follows that the integrals over the intervals $[a + (k - 1)\lambda, a + k\lambda]$ are equal and the result follows.

Geometrically, the standard regions for the intervals $[a + (k - 1)\lambda, a + k\lambda]$ are congruent.

11. Evaluate (without using the Fundamental Theorem of Calculus).

$$\int_0^{100\pi} (1 + \sin 2x)dx$$

Note: This exercise uses Exercises A5-2, No. 1(d) which requires Section A3-2(ii).

Since the integrand is periodic with period $\pi$, it follows from Number 10 that

$$\int_0^{100\pi} (1 + \sin 2x)dx = 100\int_0^{\pi} (1 + \sin 2x)dx.$$

From Exercises A5-2, Number 1(d)

$$\int_0^{\pi} \sin 2x\ dx = \frac{1 - \cos 2\pi}{2} = 0.$$

Answer: $100\pi$.

12. Prove that if $f$ is integrable on $[a,b]$ and if $f(x) \geq 0$ for all $x$ in $[a,b]$, then

$$\int_a^b f(x)dx \geq 0.$$

For every partition of $[a,b]$, $L = 0$ is ~~a~~ lower sum. Since the integral is an upper bound for all lower sums, the result follows. The result is also an immediate consequence of Theorem A5-5a.

13. Prove that if $f$ and $g$ are integrable over $[a,b]$, then

$$\left|\int_a^b \{g(x) - f(x)\}dx\right| \leq \int_a^b |g(x)|dx + \int_a^b |f(x)|dx.$$

From Number 4

$$\left|\int_a^b [g(x) - f(x)]dx\right| \leq \int_a^b |g(x) - f(x)|dx.$$

But $|g(x) - f(x)| \leq |g(x)| + |f(x)|$. It follows from Theorem A5-5a and A5-5b that

$$\int_a^b |g(x) - f(x)|dx \leq \int_a^b |g(x)|dx + \int_a^b |f(x)|dx.$$

14. Let $f$ and $g$ be integrable and suppose that $f(x) \leq g(x)$ on $[a,b]$.
(a) If the inequality $f(x) + \epsilon \leq g(x)$, for some $\epsilon > 0$, holds on any subinterval of $[a,b]$, prove the strong inequality

$$\int_a^b f(x)dx < \int_a^b g(x)dx.$$

Let $[u,v]$ be a subinterval of $[a,b]$ in which $f(x) + \epsilon < g(x)$. We have $g(x) - f(x) > \epsilon$ and by Theorem A5-5a

$$\int_u^v [g(x) - f(x)]dx \geq \int_u^v \epsilon \, dx = \epsilon \, (v - u).$$

Since $g(x) - f(x) \geq 0$ on the rest of the interval we have

$$\int_a^b [g(x) - f(x)]dx = \int_a^u [g(x) - f(x)]dx + \int_u^v [g(x) - f(x)]dx + \int_v^b [g(x) - f(x)]dx$$

$$\geq 0 + \epsilon \, (v - u) + 0$$

whence

$$\int_a^b g(x)dx - \int_a^b f(x)dx \geq \epsilon \, (v - u) > 0$$

from which the conclusion follows.

(b) If $f$ and $g$ are continuous at $x = u$ in $[a,b]$ and $f(u) < g(u)$ prove that strong inequality holds as above.

From the conditions of the problem,

$$g(u) - f(u) = \epsilon > 0.$$

Also, continuity of $f$ and $g$ implies that there is some neighborhood of $u$ in which

$$|f(x) - f(u)| < \frac{\epsilon}{4}$$

and

$$|g(x) - g(u)| < \frac{\epsilon}{4}.$$

Combining, we obtain

$$g(x) - f(x) > \frac{\epsilon}{2}$$

in some neighborhood of $u$. The result follows from part (a).

15. If functions $f$ and $g$ are integrable, and $f(x) \le h(x) \le g(x)$ on $[a,b]$, does it follow that

$$\int_a^b f(x)dx \le \int_a^b h(x)dx \le \int_a^b g(x)dx?$$

Illustrate by an example.

No! The function $h$ may not be integrable. For example, take $f(x) = -1$, $g(x) = 1$,

$$h(x) = \begin{cases} 1, & x \text{ rational} \\ -1, & x \text{ irrational.} \end{cases}$$

On every interval $\max\{h(x)\} = 1$, $\min\{h(x)\} = -1$, hence for every upper sum, $U \ge (b - a)$, and for every lower sum, $L \le -(b - a)$. Thus $h$ is not integrable by Theorem A5-4a.

16. (a) Prove the Mean Value Theorem of integral calculus: If $f$ is continuous and integrable on $[a,b]$, then there exists a value $u$ in the open interval $(a,b)$ such that

$$\int_a^b f(x)dx = f(u)(b - a).$$

By the Extreme Value Theorem $f(x)$ takes on a maximum value $M$ and a minimum value $m$ in $[a,b]$. Since

$$m \le f(x) \le M$$

on $[a,b]$ we have from Theorem A5-5a

$$\int_a^b m\,dx \le \int_a^b f(x)dx \le \int_a^b M\,dx$$

whence

$$m \le \frac{1}{b - a}\int_a^b f(x)dx \le M.$$

Since $f(x)$ takes on every value between $m$ and $M$ (Intermediate Value Theorem, there is a value $u$ in $[a,b]$ for which

$$f(u) = \frac{1}{b - a}\int_a^b f(x)dx.$$

(b) Show that the value $f(u)$ in (a) satisfies

$$f(u) = \lim_{h \sim 0} \frac{f_0 + f_1 + \dots + f_n}{n + 1}$$

where $h = \frac{(b - a)}{n}$ and $f_k = f(a + kh)$ for $k = 0, 1, 2, \dots,$ $n$. Thus $f(u)$ can be interpreted as an extension of the idea of mean or arithmetic average to the values of a function on an interval.

The expression for the integral as a limit of Riemann sums is

$$\int_a^b f(x)dx = \lim_{h \sim 0} \sum_{k=1}^{n} f_k h = \lim_{h \sim 0} \sum_{k=0}^{n} f_k h$$

$$= \lim_{h \sim 0} (n + 1)h \left\{ \frac{\sum_{k=0}^{n} f_k}{n + 1} \right\}$$

$$= \lim_{h \sim 0} (b - a + h) \left\{ \frac{\sum_{k=0}^{n} f_k}{n + 1} \right\}$$

$$= \lim_{h \sim 0} (b - a + h) \lim_{h \sim 0} \left\{ \frac{\sum_{k=0}^{n} f_k}{n + 1} \right\}$$

from which the result follows (provided

$$\lim_{h \sim 0} \left\{ \frac{\sum_{k=0}^{n} f_k}{n + 1} \right\}$$

exists.) Existence is a consequence of the fact that if $\lim_{h \sim 0} pq$ exists and $\lim_{h \sim 0} p$ exists but $\lim_{h \sim 0} p \neq 0$ then $\lim_{h \sim 0} q$ exists and

$$\lim_{h \sim 0} q = \frac{\lim_{h \sim 0} pq}{\lim_{h \sim 0} p}.$$

(The point need not be brought up unless the issue of existence is raised in class.)

17. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + \frac{a_n}{1} = 0$, show that

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

has at least one real root in $(0,1)$.

Set $f(x) = \sum_{k=0}^{n} a_k x^{n-k}$. By the Mean Value Theorem (No. 16a) there is a point $u$ in $(0,1)$ for which,

$$f(u) = \int_0^1 f(x)dx$$

$$= \frac{a_0x^{n+1}}{n+1} + \frac{a_1x^n}{n} + \ldots + \frac{a_nx}{1}\Big|_0^1$$

$$= \frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_n}{1}$$

$$= 0.$$

18. (a) Prove that if $f(x)$ is integrable on $[a,b]$, then $|f(x)|$ is integrable on $[a,b]$.

For each positive $\epsilon$ there exist a partition $\sigma = \{x_1,x_2,x_3,\ldots,x_n\}$, an upper sum $U$, and a lower sum $L$ over $\sigma$ for which

$$U - L = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \epsilon,$$

where $m_k \leq f(x) \leq M_k$ on $[x_{k-1},x_k]$. We shall describe upper and lower bounds $M_k^*$ and $m_k^*$ for $|f(x)|$ on each interval $[x_{k-1},x_k]$. If $m_k \geq 0$, then $|f(x)| = f(x)$ and we take $M_k^* = M_k$ and $m_k^* = m_k$ on $[x_{k-1},x_k]$. If $M_k \leq 0$, then $|f(x)| = -f(x)$ and we take $M_k^* = -m_k$ and $m_k^* = -M_k$; whence

$$M_k^* - m_k^* = -m_k - (-M_k) = M_k - m_k.$$

If $M_k \geq 0$ and $m_k < 0$ we consider two cases:

(i) $|m_k| \leq M_k$. Taking $M_k^* = M_k$ and $m_k^* = 0$, we have

$$M_k^* - m_k^* = M_k < M_k + |m_k| \leq M_k - m_k.$$

(ii) $|m_k| > M_k$. Taking $M_k^* = |m_k|$ and $m_k^* = 0$, we have

$$M_k^* - m_k^* = |m_k| \leq |m_k| + M_k \leq M_k - m_k.$$

In every case

$$U^* - L^* \leq U - L < \epsilon$$

where $U^*$ and $L^*$ are the upper and lower sums for $|f(x)|$ constructed by use of the bounds $M_k^*$ and $m_k^*$.

Alternate Solution

Define $f^+$ and $f^-$ by

$$f^+(x) = \begin{cases} f(x), & \text{if } f(x) \geq 0, \\ 0, & \text{if } f(x) \leq 0, \end{cases} \qquad f^-(x) = \begin{cases} -f(x), & \text{if } f(x) \leq 0, \\ 0, & \text{if } f(x) \geq 0. \end{cases}$$

Since $|f| = f^+ + f^-$, it is necessary only to show that $f^+$ and $f^-$ are integrable in order to prove that $|f|$ is integrable. Now, for each positive $\epsilon$, there exist a partition $\sigma$, an upper sum $U$, and a lower sum $L$ for $f$ over $\sigma$ such that, in the notation of the text,

$$U - L = \sum (M_k - m_k)(x_k - x_{k-1}) < \epsilon.$$

On each subinterval $[x_{k-1}, x_k]$ choose upper and lower bounds $M^+$ and $m^+$ for $f^+$ as follows. If $f(x) \geq 0$ for at least one point $x$ in $[x_{k-1}, x_k]$ take $M_k^+ = M_k$ and $m_k^+ = m_k$; if $f(x) < 0$ everywhere on the interval, take $M_k^+ = m_k^+ = 0$. In either case

$$M_k^+ - m_k^+ \leq M_k - m_k.$$

Similarly, for $f^-$ the corresponding upper and lower bounds satisfy

$$M_k^- - m_k^- \leq M_k - m_k.$$

It follows for the corresponding upper and lower sums that

$$U^+ - L^+ < \epsilon, \quad U^- - L^- < \epsilon.$$

Consequently, $f^+$ and $f^-$ are integrable and so also is $|f|$.

19. Suppose

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational.} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

Show that if $U$ and $L$ are upper and lower sums for a partition of $[0,1]$ then $U \geq 1$ and $L \leq -1$. Is $f$ integrable on $[0,1]$?

See Number 15.

Note that $|f(x)| = 1$ on $[0,1]$ and that

$$\int_0^1 |f(x)|dx = 1.$$

Thus, the fact that $|f(x)|$ is integrable on $[a,b]$ does not imply that $f(x)$ is integrable on $[a,b]$.

20. If $f$ and $g$ are integrable on $[a,b]$ then both $\max\{f,g\}$ and $\min\{f,g\}$ are also integrable on $[a,b]$.

Use Number 19 and the results of Exercises A1-3, Number 7:

$$\max\{f,g\} = \frac{1}{2}(f + g + |f - g|),$$

$$\min\{f,g\} = \frac{1}{2}(f + g - |f - g|);$$

the result follows from Lemma A5-5b.

<u>Alternate Solution</u>

There exists a partition $\sigma$ and upper and lower sums $U_1$ and $L_1$ for $f$, $U_2$ and $L_2$ for $g$, over $\sigma$ such that both

$$U_1 - L_1 < \varepsilon \text{ and } U_2 - L_2 < \varepsilon.$$

(For example, if $\sigma_1$ is such a partition for $f$, $\sigma_2$ for $g$, take $\sigma = \sigma_1 \cup \sigma_2$.) Let $M_k'$, $m_k'$ denote the bounds for $f$ in the expressions for $U_1$ and $L_1$, $M_k''$ and $m_k''$ the corresponding bounds for $g$. Set $\Phi = \max\{f,g\}$. On each interval $[x_{k-1},x_k]$ choose as the upper bound for $\Phi$, $M_k = \max\{M_k',M_k''\}$; as the lower bound, choose

$$m_k = \begin{cases} m_k', & \text{if } M_k = M_k', \\ m_k'', & \text{otherwise.} \end{cases}$$

With this choice,

$$M_k - m_k \le \max\{M_k' - m_k', M_k'' - m_k''\}$$
$$\le (M_k' - m_k') + (M_k'' - m_k'').$$

Form the appropriate upper and lower sums U and L for $\Phi$ to obtain

$$U - L \le (U_1 - L_1) + (U_2 - L_2) < 2\epsilon$$

from which the conclusion follows.

To obtain the result for $\min\{f,g\}$ observe that $\min\{f,g\} = -\max\{-f,-g\}$.

Finally, since $|f| = \max\{f,-f\}$, this proof also serves to demonstrate Number 21.

21. Let f and g be bounded and integrable on $[a,b]$.
(a) Prove $f \cdot g$ is integrable on $[a,b]$.

First observe that it is sufficient to prove the result when f and g are positive. This is true from the boundedness of f and g, because there exist constants $c_1$ and $c_2$ such that $\phi = f + c_1$ and $\psi = g + c_2$ are positive. If the result is true for positive functions then $\phi \cdot \psi = f \cdot g + c_1 g + c_2 f + c_1 c_2$ is integrable, hence $f \cdot g$ is integrable.

Now let $\sigma$ be a partition, $M_k'$ and $m_k'$ upper and lower bounds for f, $M_k''$ and $m_k''$ for g, on the subinterval $[x_{k-1}, x_k]$ of the partition, and let $U'$, $L'$ and $U''$, $L''$ be the corresponding upper and lower sums. For a sufficiently fine partition and appropriate choice of bounds,

$$U' - L' < \epsilon \text{ and } U'' - L'' < \epsilon.$$

Now, choose the upper and lower bounds $M_k = M_k' M_k''$, $m_k = m_k' m_k''$ for $f \cdot g$. Then

$$M_k - m_k = M_k'(M_k'' - m_k'') + m_k''(M_k' - m_k')$$
$$\le A(M_k'' - m_k'') + B(M_k' - m_k')$$

where A and B are overall upper bounds for f and g, respectively. Form the appropriate upper and lower sums for $f \cdot g$ over $\sigma$ to obtain

$$U - L \le A(U' - L') + B(U'' - L'')$$
$$< \epsilon(A + B),$$

from which it follows that $f \cdot g$ is integrable.

(b) If $g$ is bounded away from zero, then $\frac{f}{g}$ is integrable over $[a,b]$.

For the proof, it is sufficient to prove $\frac{1}{g}$ is integrable and then to apply Part (a). Now suppose $|g(x)| \geq c > 0$. Let $U$ and $L$ be upper and lower sums for $g$ over $\sigma$ such that $U - L < \epsilon$, and let $M_k$, $m_k$ denote upper and lower bounds for $g$ on the subinterval $I_k = [x_{k-1}, x_k]$. Choose upper and lower bounds $M_k^*$ and $m_k^*$ for $\frac{1}{g}$ as follows. If $m_k \geq c$ or $M_k \leq -c$, then take $M_k^* = \frac{1}{m_k}$ and $m_k^* = \frac{1}{M_k}$. In this case, $M_k^* - m_k^* = \frac{M_k - m_k}{m_k M_k} \leq \frac{M_k - m_k}{c^2}$. If $m_k \leq -c$ and $M_k \geq c$, then take $m_k^* = -\frac{1}{c}$ and $M_k^* = \frac{1}{c}$. In this case use $1 \leq \frac{M_k}{c}$ and $1 \leq -\frac{m_k}{c}$ to obtain

$M_k^* - m_k^* = \frac{1}{c} + \frac{1}{c} \leq \frac{M_k}{c^2} - \frac{m_k}{c^2} \leq \frac{M_k - m_k}{c^2}$. For the corresponding upper and lower sums $U^*$ and $L^*$, then

$$U^* - L^* \leq \frac{U - L}{c^2} < \frac{\epsilon}{c^2}.$$

22. If $f$ and $g$ are bounded and integrable, then $\int_a^b \left(\alpha f(x) + \beta g(x)\right)^2 dx$ exists and is greater than or equal to 0 for all constant $\alpha$ and $\beta$. Show from this that

$$\int_a^b f(x)^2 dx \int_a^b g(x)^2 dx \geq \left\{ \int_a^b f(x)g(x)\,dx \right\}^2$$

with equality if and only if (for $f$ and $g$ continuous) $f = cg$ on $[a,b]$ for some constant $c$.

Consider the inequality

$$\int_a^b \left(f(x) + tg(x)\right)^2 dx = \int_a^b f(x)^2 dx + 2t \int_a^b f(x)g(x)dx + t^2 \int_a^b g(x)^2 dx$$
$$\geq 0.$$

An inequality of the form $At^2 + 2Bt + C > 0$ holds for all $t$ if and only if $B^2 - AC \leq 0$, but that is precisely Buniakowsky's inequality in this case.

If $f = cg$ then equality obviously holds. If equality holds $B^2 - AC = 0$, then for some choice of $t$, say $t = -c$

$$\int_a^b [f(x) - cg(x)]^2 dx = 0.$$

Now $f - cg$ must be identically zero, for if there were any point where $f(x_0) - cg(x_0) \neq 0$, then $[f(x_0) - cg(x_0)]^2 > 0$ and from the continuity of $f$ and $g$ there would be an interval containing $x_0$ where the integrand has a positive lower bound. In that case the integral would have to be positive, not zero as required. Consequently, $f = cg$ is the only possibility. (Note that the proof requires the continuity of $f$ and $g$ at only one point.)

23. If $f$ is integrable and its graph is convex on the interval $[0,a]$, show that

$$\int_0^a f(x)dx \geq a\, f(\tfrac{a}{2}).$$

Interpret geometrically.

The graph of $f$ lies above its tangent at $\frac{a}{2}$:

$$f(x) \geq f(\tfrac{a}{2}) + f'(\tfrac{a}{2})(x - \tfrac{a}{2})$$

(if $f$ is not differentiable there still exists a "line of support" at $\frac{a}{2}$ and $f'(\frac{a}{2})$ would be replaced by the slope $m$ of a line of support). Then

$$\int_0^a f(x)dx \geq \int_0^a [f(\tfrac{a}{2}) + f'(\tfrac{a}{2})(x - \tfrac{a}{2})]dx$$

$$\geq \left[f(\tfrac{a}{2})x + f'(\tfrac{a}{2})\left(\frac{x^2}{2} - \frac{ax}{2}\right)\right]\Bigg|_0^a$$

$$\geq a\, f(\tfrac{a}{2}).$$

For $f$ positive, the geometrical intepretation is that the area of the standard region under the graph of $f$ is greater than the area under the tangent taken at the midpoint of the base interval.

24. Show that

$$\sqrt{(a^2 + \tfrac{1}{3})(b^2 + \tfrac{1}{3})} \geq \int_0^1 \sqrt{(x^2 + a^2)(x^2 + b^2)}\, dx$$

Set $f(x) = \sqrt{x^2 + a^2}$, $g(x) = \sqrt{x^2 + b^2}$ in the Buniakowsky inequality (No. 22).

25. Show that

(a) $\frac{1}{2} + \frac{3\sqrt{2}}{8} < \int_0^1 \sqrt{1 + x^3}\, dx < \frac{\sqrt{5}}{2}$.

Take $f(x) = 1$, $g(x) = \sqrt{1 + x^3}$ in the Buniakowsky inequality to obtain the upper estimate. For the lower estimate, use

$$\int_0^1 \sqrt{1 + x^3}\, dx = \int_0^{1/2} \sqrt{1 + x^3}\, dx + \int_{1/2}^1 \sqrt{1 + x^3}\, dx$$

$$> \int_0^{1/2} 1\, dx + \int_{1/2}^1 \sqrt{1 + \tfrac{1}{8}}\, dx.$$

(b) Show that $\frac{1}{2} + \frac{\sqrt{2}}{3} > \int_0^1 \frac{dx}{\sqrt{1 + x^3}} > \frac{2\sqrt{5}}{5}$.

For the upper estimate, use

$$\int_0^1 \frac{dx}{\sqrt{1 + x^3}} = \int_0^{1/2} \frac{dx}{\sqrt{1 + x^3}} + \int_{1/2}^1 \frac{dx}{\sqrt{1 + x^3}}$$

$$< \int_0^1 1\, dx + \int_{1/2}^1 \frac{dx}{\sqrt{1 + \frac{1}{8}}}.$$

For the lower estimate use the Buniakowsky inequality:

$$\int_0^1 \sqrt{1 + x^3}\, dx \int_0^1 \frac{dx}{\sqrt{1 + x^3}} > 1,$$

hence, from Part (a),

$$\frac{\sqrt{5}}{2} \int_0^1 \frac{dx}{\sqrt{1 + x^3}} > 1.$$

26. Find a continuously differentiable function $F$ in $[0,1]$ such that.

(a) $F(0) = 0, \quad F(1) = a,$

(b) $\displaystyle\int_0^1 F(x)^2 dx = \frac{a^2}{3},$

(c) $\displaystyle\int_0^1 F'(x)^2 dx$ is a minimum.

Take $f(x) = 1$ and $g(x) = F'(x)$ in the Buniakowsky inequality (No. 22):

$$\int_0^1 1\,dx \int_0^1 F'(x)^2\,dx \geq \left\{\int_0^1 F'(x)dx\right\}^2$$

$$\geq \{F(1) - F(0)\}^2$$

$$\geq a^2.$$

Equality holds when $g = cf$ for constant $c$. Thus $F'(x) = c$, hence $F(x) = cx + d$. From condition (a), $F(x) = ax$; condition (b) is automatically satisfied and is therefore redundent. Condition (c) is satisfied since equality is ashieved in the inequality above.

Teacher's Commentary

Appendix 6

# INEQUALITIES AND LIMITS

## TC A6-1. Absolute Value and Inequality

In Section A1-1 (footnote), we define $|a|$ as $\sqrt{a^2}$. This definition has the virtue of emphasizing the positivity of the square root. It also helps to prevent the error of writing $\sqrt{a^2} = a$ in case $a < 0$. This error leads to the amusing "proof":

$$1 = \sqrt{1} = \sqrt{(-1)^2} = -1$$

thus

$$1 = -1.$$

We note that the form $\sqrt{a^2}$ lends itself, more conveniently, to mathematical manipulation.

### Solutions Exercises A6-1

1. Find the absolute value of the following numbers.

   (a) $-1.75$ $\qquad$ $1.75$

   (b) $-\frac{\pi}{4}$ $\qquad$ $\frac{\pi}{4}$

   (c) $\sin(-\frac{\pi}{4})$ $\qquad$ $\frac{\sqrt{2}}{2}$

   (d) $\cos(-\frac{\pi}{2})$ $\qquad$ $0$

2. (a) For what real numbers $x$ does $\sqrt{x^2} = -x$?

   $x \leq 0$

   (b) For what real numbers $x$ does $|1 - x| = x - 1$?

   $x \geq 1$

3. Solve the equations:

(a) $|3 - x| = 1$ — $x = 2$ or $x = 4$.

(b) $|4x + 3| = 1$ — $x = -\frac{1}{2}$ or $x = -1$.

(c) $|x + 2| = x$ — Either $x + 2 \geq 0$ or $x + 2 < 0$, then $x + 2 = x$ or $-(x + 2) = x$. Thus there are no solutions.

(d) $|x + 1| = |x - 3|$ — The only solution is $x = 1$.

(e) $|2x + 5| + |5x + 2| = 0$ — There are no solutions.

(f) $|2x + 3| = |5 - x|$ — $x = -8$ or $x = \frac{2}{3}$.

(g) $2|3x + 4| + |x - 2| = 1 + |3 + x|$ — There are no solutions.

4. For what values of $x$ is each of the following true? (Express your answer in terms of inequalities satisfied by $x$.)

(a) $|x| \leq 0$ — $x = 0$

(b) $|x| \neq x$ — $x < 0$

(c) $|x| < 3$ — $-3 < x < 3$

(d) $|x - 6| \leq 1$ — $5 \leq x \leq 7$

(e) $|x - 3| > 2$ — $x < 1$ or $x > 5$

(f) $|2x - 3| < 1$ — $1 < x < 2$

(g) $|x - a| < a$ — $0 < x < 2a$

(h) $|x^2 - 3| < 1$ — $\sqrt{2} < x < 2$ or $-2 < x < -\sqrt{2}$

(i) $|(x - 2)(x - 3)| > 2$ — $x < 1$ or $x > 4$

(j) $|x - 1| > |x - 3|$ — $x > 2$

(k) $|x - 5| + 1 = |x + 5|$ — $x = \frac{1}{2}$

(ℓ) $|x - 1| + |x - 2| = 1$ $\qquad 1 \leq x \leq 2$

(m) $|x^2 - a^2| > 0$ $\qquad x \neq \pm a$

(n) $|x - a| < \delta$ $\qquad a - \delta < x < a + \delta$

(o) $0 < |x - a| < \delta$ $\qquad a - \delta < x < a$ or $a < x < a + \delta$

(p) $|x - 1| < 2$ and $|x + 1| < \frac{3}{2}$ $\qquad -1 < x < \frac{1}{2}$

(q) $|x - 1| < 2$ and $|2x - 1| < \frac{3}{2}$ $\qquad -\frac{1}{4} < x < \frac{5}{4}$

(r) $|x + y| = |x| + |y|$, for all $y$ $\qquad x = 0$

(s) $|\sin x| = 0$ $\qquad x = n\pi$, $n$, an integer

(t) $|\sin x| > \frac{\sqrt{2}}{2}$ $\qquad \frac{\pi}{4} + \pi n < x < \frac{3\pi}{4} + \pi n$

(u) $|1 - \frac{1}{x}| < 1$ $\qquad x > \frac{1}{2}$

(v) $\sqrt{|x|} > \frac{1}{2}$ $\qquad x < -\frac{1}{4}$ or $x > \frac{1}{4}$

5. Sketch the graphs of the following equations:

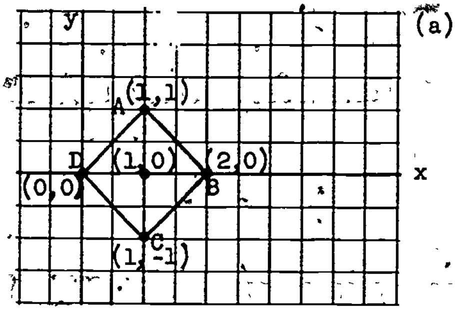
(a) $|x - 1| + |y| = 1$

For $x \geq 1$, $y > 0$, then $x - 1 + y = 1$ or $x + y = 2$, line AB.

For $x \geq 1$, $y < 0$, then $x - 1 - y = 1$ or $x - y = 2$, line BC.

For $x < 1$, $y < 0$, then $-x + 1 - y = 1$ or $x + y = 0$, line CD.

For $x < 1$, $y \geq 0$, then $-x + 1 + y = 1$, or $y = x$, line DA.

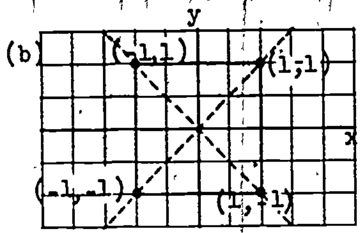


(b) $|x + y| + |x - y| = 2$.

Resolves into 4 parts:

$x = \pm 1$ and $y = \pm 1$

where $|x| \leq 1$ and $|y| \leq 1$

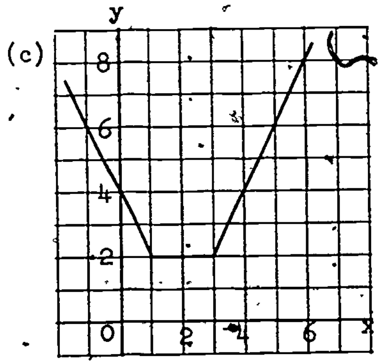



(c) $y = |x - 1| + |x - 3|$

For $x < 1$, then $y = -2x + 4$.

For $1 \leq x \leq w$, then $y = 2$.

For $x > 3$, then $y = 2x - 4$.



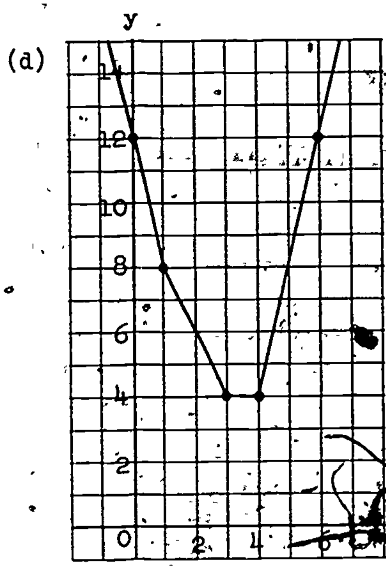

(d) $y = |x - 1| + |x - 3| + 2|x - 4|$

For $x < 1$, then $y = -4x + 12$.

For $1 \leq x < 3$, then $y = -2x + 10$.

For $3 \leq x < 4$, then $y = 4$.

For $4 \leq x$, then $y = 4x - 12$.



(e) $y = |x - 1| + |x - 3| + 2|x - 4| + 3|x - 5|$

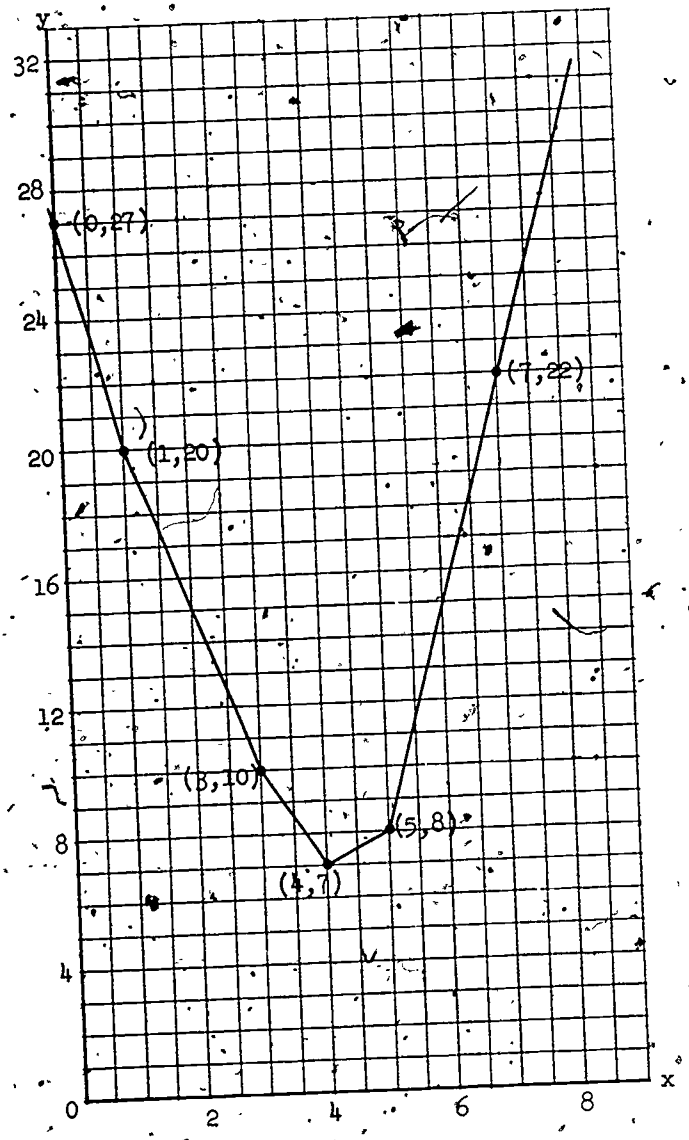
For $x < 1$, then $y = -7x + 27$.

For $1 \leq x < 3$, then $y = -5x + 25$.

For $3 \leq x < 4$, then $y = -3x + 19$.

For $4 \leq x < 5$, then $y = x + 3$.

For $5 \leq x$, then $y = 7x - 27$.



6. (a) Show that if $a > b > 0$, then $\frac{ab}{a + b} < b$.

$$0 < b \longrightarrow a < a + b$$
$$\longrightarrow ab < b(a + b)$$
$$\Longrightarrow \frac{ab}{a + b} < b, \quad \text{since} \quad a + b > 0.$$

(b) Thus, show that for positive numbers $a$ and $b$, the condition $\delta \leq \min\{a,b\}$ is satisfied by $\delta = \frac{ab}{a + b}$.

For $a \neq b$ the result follows from part (a). For $a = b$, $\delta = \frac{a}{2} < a$.

7. (a) Show for positive $a$, $b$ that $\frac{a + b}{2} < \max\{a,b\}$ if $a \neq b$.

$$\frac{a + b}{2} < \frac{\max\{a,b\} + \max\{a,b\}}{2} \leq \max\{a,b\}$$

(b) Prove for all $a,b$ that

$$\max\{a,b\} = \frac{1}{2}(a + b + |a - b|)$$

(c) $$\min\{a,b\} = \frac{1}{2}(a + b - |a - b|)$$

Assume, without loss of generality, that $a \geq b$,

then $\quad \max\{a,b\} = a = \frac{1}{2}(a + b + a - b)$,

and $\quad \min\{a,b\} = b = \frac{1}{2}(a + b - (a-b))$.

8. Show that

$$\max\{a,b\} + \max\{c,d\} \geq \max\{a + c, b + d\}.$$

From Number 7(b)

$$\max\{a,b\} + \max\{c,d\} = \frac{1}{2}(a + b + c + d + |a - b| + |c - d|),$$
$$\max\{a + c, b + d\} = \frac{1}{2}(a + b + c + d + |a + c - (b + d)|)$$
$$= \frac{1}{2}(a + b + c + d + |(a - b) + (c - d)|).$$

The result follows at once.

9. Show that if $ab \geq 0$, then $ab \geq \min\{a^2, b^2\}$.

$$ab = |a|\,|b| \geq (\min\{|a|,|b|\})^2 = \min\{a^2, b^2\}.$$

10. Show that if $a = \max\{a,b,c\}$, then $-a = \min\{-a,-b,-c\}$.

If $a = \max\{a,b,c\}$, $a \geq b$, $a \geq c$,

then $-a \leq -b$ and $-a \leq -c$.

So, $-a = \min\{-a,-b,-c\}$.

11. Denote $\min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right\}$ by $\min\limits_r\left\{\frac{a_r}{b_r}\right\}$ and similarly for max.

If $b_r > 0$, $r = 1, 2, \ldots, n$, prove that

$$\min_r\left\{\frac{a_r}{b_r}\right\} \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq \max_r\left\{\frac{a_r}{b_r}\right\}.$$

Denote $\min\limits_r\left\{\frac{a_r}{b_r}\right\}$ by $\frac{a_k}{b_k}$, $1 \leq k \leq n$, and

$\max\limits_r\left\{\frac{a_r}{b_r}\right\}$ by $\frac{a_e}{b_e}$, $1 \leq e \leq n$.

Then, $\frac{a_k}{b_k} \leq \frac{a_r}{b_r}$, $r = 1, 2, \ldots, n$.

Or, $a_k b_r \leq b_k a_r$, for all $r$. Adding,

$$a_k b_1 + a_k b_2 + \ldots + a_k b_n \leq b_k a_1 + b_k a_2 + \ldots + b_k a_n.$$

Factoring and dividing, we obtain

$$\min\left\{\frac{a_r}{b_r}\right\} = \frac{a_k}{b_k} \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + a_n}.$$

In the same way the remaining inequality may be obtained.

12. Prove that

$$\frac{1}{n} \leq \frac{1 + 2 + \ldots + n}{(n)^2 + (n-1)^2 + \ldots + 2^2 + 1^2} \leq 1 \text{ for } n = 1, 2, 3, \ldots, n.$$

Use the inequality obtained in Number 11, with

$$\frac{a_r}{b_r} = \frac{r}{r^2} \text{ or } \frac{1}{r}.$$

13. (a) Prove directly from the properties of order for $\epsilon > 0$ that if $-\epsilon \leq x \leq \epsilon$ then $|x| \leq \epsilon$. Conversely, if $|x| \leq \epsilon$ then $-\epsilon \leq x \leq \epsilon$.

Suppose $-\epsilon \leq x \leq \epsilon$. If $0 < x$, $|x| = x \leq \epsilon$. If $x < 0$, $|x| = -x$. But $-\epsilon \leq x$ implies $-x \leq \epsilon$. So, $-x = |x| \leq \epsilon$.

Conversely, suppose $|x| \leq \epsilon$. If $0 \leq x$, $|x| = x$, so $-\epsilon < 0 \leq x \leq \epsilon$, thus $-\epsilon \leq x \leq \epsilon$. Similarly for $x < 0$.

(b) Prove that if $x$ is an element of an ordered field and if $|x| < \epsilon$ for all positive values $\epsilon$, then $x = 0$.

If $x \neq 0$, take $\epsilon = |x|$. We then have the contradictory statements $|x| = |x|$ and $|x| < |x|$.

14. (a) Prove that $|ab| = |a| \cdot |b|$.

Just consider the three cases.

$$ab > 0, \quad ab = 0, \quad ab < 0.$$

(b) Prove that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $b \neq 0$.

From part(a) we have

$$\left|\frac{a}{b}\right| = \left|a\left(\frac{1}{b}\right)\right| = |a|\left|\frac{1}{b}\right|,$$

and

$$|b| \cdot \left|\frac{1}{b}\right| = \left|\frac{b}{b}\right| = |1| = 1.$$

Hence $$\left|\frac{1}{b}\right| = \frac{1}{|b|}.$$

Therefore $$|a|\left|\frac{1}{b}\right| = |a| \cdot \frac{1}{|b|} = \frac{|a|}{|b|}.$$

15. Prove that $|x - y| \leq |x| + |y|$.

In $|a + b| \leq |a| + |b|$, set $a = x$, $b = -y$.

16. Under what conditions do the equality signs hold for
$$\big||a| - |b|\big| \leq |a + b| \leq |a| + |b|?$$

Equality occurs only if $a = b = 0$.

17. If $0 < x < 1$, we can multiply both sides of the inequality $x < 1$ by $x$ to obtain $x^2 < x$ (and, similarly, we can show that $x^3 < x^2$, $x^4 < x^3$, and so on). Use this result to show that if $0 < |x| < 1$, then $|x^2 + 2x| < 3|x|$.

$$|x^2 + 2x| \leq |x^2| + |2x| \leq |x| + |2x| = 3|x|,$$
$$(|x^2| = |x|^2 < |x| \text{ since } 0 < |x| < 1).$$

18. Prove the following inequalities

(a) $x + \frac{1}{x} \geq 2$, $x > 0$.

Since $(x - 1)^2 \geq 0$, we have

$$x^2 - 2x + 1 \geq 0 \quad \text{or} \quad x^2 + 1 \geq 2x.$$

Since $\frac{1}{x} > 0$, we obtain $x + \frac{1}{x} \geq 2$.

(b) $x + \frac{1}{x} \leq -2$, $x < 0$.

$(x + 1)^2 \geq 0$. So $x^2 + 2x + 1 \geq 0$, or $x^2 + 1 \geq -2x$.

Since $x < 0$, $\frac{1}{x} < 0$, so $\frac{1}{x}(x^2 + 1) \leq \frac{1}{x}(-2x)$ or $x + \frac{1}{x} \leq -2$.

(c) $|x + \frac{1}{x}| \geq 2$, $x \neq 0$.

From (a) we have $x + \frac{1}{x} \geq 2$ for $x > 0$

or $|x + \frac{1}{x}| \geq 2$ for $x > 0$.

From (b) we have $-(x + \frac{1}{x}) \geq 2$ for $x < 0$

or $-(x + \frac{1}{x}) = |x + \frac{1}{x}| \geq 2$ for $x < 0$.

Thus $|x + \frac{1}{x}| \geq 2$ for $x \neq 0$.

19. Prove: $x^2 \geq x|x|$ for all real $x$.

If $x \geq 0$, $x = |x|$, and $x^2 = x|x|$.

If $x < 0$, $x|x| < 0 < x^2$.

20. Show that if $|x - a| < \frac{|a|}{2}$, then $\frac{|a|}{2} < |x| < \frac{3|a|}{2}$ for all $a \neq 0$.

Using the inequalities of 13(a), we obtain

$$|x - a| < \frac{|a|}{2} \Longrightarrow -\frac{|a|}{2} < x - a < \frac{|a|}{2}$$

$$\Longrightarrow \frac{|a|}{2} < x < \frac{3|a|}{2}.$$

21. Prove for positive a and b, where $a \neq b$, that

$$\frac{|b - a|^2}{4(a + b)} < \frac{a + b}{2} - \sqrt{ab} < \frac{|b - a|^2}{8\sqrt{ab}} .$$

To avoid $\sqrt{\phantom{x}}$, let $a = m^2$, $b = n^2$, then we have to show that

$$\frac{|m^2 - n^2|^2}{4(m^2 + n^2)} < \frac{m^2 + n^2 - 2mn}{2} < \frac{|m^2 - n^2|^2}{8mn}$$

or

$$\frac{(m + n)^2}{2(m^2 + n^2)} < 1 < \frac{(m + n)^2}{4mn} .$$

Both of these inequalities are equivalent to $(m - n)^2 > 0$.
By way of example we show one of the equivalences:

$$\begin{aligned}(m - n)^2 > 0 &\longrightarrow m^2 + n^2 > 2mn \\ &\longrightarrow 2(m^2 + n^2) > (m + n)^2 \\ &\longrightarrow 1 > \frac{(m + n)^2}{2(m^2 + n^2)} .\end{aligned}$$

## TC A6-2. Definition of Limit of a Function

In some texts, the idea of limit often is expressed in words like these: "If, as $x$ gets closer and closer to $a$, the values of $f(x)$ tend to the value $L$, then we call $L$ the limit of $f(x)$ as $x$ approaches $a$." The difficulty with this formulation, apart from the vagueness of the words, "gets closer and closer to," "tend to," is that it suggests the false notion that if $x_2$ is closer to $a$ than is $x_1$, then $f(x_2)$ is closer to $L$ than is $f(x_1)$.

Example TC A6-2. Consider

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

We have $\lim_{x \to 0} f(x) = 0$. Let $x_1 = \frac{1}{2n\pi}$ and $x_2 = \frac{2}{\pi(1 + 4n)}$ ($n$, a non-zero integer). Then $|x_2| < |x_1|$ but $|f(x_2)| > |f(x_1)|$ since $f(x_2) = \frac{2}{\pi(1 + 4n)}$ and $f(x_1) = 0$.

The above description of limit gives no clear idea of just how to verify that $L$ is the limit of $f$ as $x$ approaches $a$ in any particular case. We are compelled to give a definition which yields a clear-cut method of verification.

Quite early in our discussion we refer to Appendix 1-4 for an explanation of open and closed intervals. These ideas are essential to the material in this and succeeding sections. In the exercises, substantial use is made of ideas relating to the order properties of real numbers and absolute value: the student is expected to apply basic inequality theorems. The objective is to develop computational facility with absolute value as a background for proving facts about limits. As a lead into Section 3-3 we feel that it would be informative for the student to be given some numerical values for $\epsilon$ and be required to determine a $\delta$ sufficient to control the error (see, for example, Exercises 3-2, No. 11).

## Solutions Exercises A6-2

The theorems of Section A6-1 provide the basis for the following arguments. In the general application of the transitive property we use the strong inequality in the conclusion since a strong inequality appears at least once in the chain of reasoning. We also make extended use of the inequalities

$$\big||a| - |b|\big| \le |a + b| \le |a| + |b|.$$

1. Show that if $0 < |x - a| < 1$, then $|x + 2a| < 1 + 3|a|$.

If $a < |x - a| < 1$, then

$$\begin{aligned} |x + 2a| &= |(x - a) + 3a| \\ &\le |x - a| + |3a| \\ &\le |x - a| + 3|a| \\ &< 1 + 3|a|. \end{aligned}$$

2. Show that if $0 < |x - a| < 1$, then $|x^3 - a^3| < (3|a^2| + 3|a| + 1)|x - a|$.

If $0 < |x - a| < 1$, then

$$\begin{aligned} |x^3 - a^3| &= |(x - a)(x^2 + ax + a^2)| \\ &= |x - a| \cdot |((x - a) + a)^2 + a((x - a) + a) + a^2| \\ &= |x - a| \cdot |(x - a^2) + 3a(x - a) + 3a^2| \\ &\le |x - a| \cdot [(x - a)^2 + 3|a| \cdot |x - a| + 3a^2] \\ &< 1 + 3|a| + 3a^2. \end{aligned}$$

3. Show that if $0 < |x - 2| < 1$, then $\frac{1}{|x - 4|} < 1$.

Hint: If $|x - 4| > 1$, then $\frac{1}{|x - 4|} < 1$.

We have $|x - 4| = |(x - 2) - 2|$ whence

$$|-2| - |x - 2| \le |x - 4| \le |x - 2| + |-2|.$$

Thus, if $0 < |x - 2| < 1$,

$$2 - 1 < |x - 4| < 1 + 2$$

or

$$1 < |x - 4| < 3$$

and

$$\frac{1}{|x - 4|} < 1.$$

4. Show that if $|x - a| < \frac{|a|}{2}$, then $\frac{1}{x^2} < \frac{4}{a^2}$.

We have $|x| = |(x - a) + a|$, so that

$$|a| - |x - a| \le |x| \le |x - a| + |a|.$$

Thus, if $|x - a| < \frac{|a|}{2}$,

$$|a| - \frac{|a|}{2} < |x| < \frac{|a|}{2} + |a|$$

or

$$\frac{|a|}{2} < |x| < \frac{3|a|}{2}$$

whence

$$\frac{a^2}{4} < x^2 < \frac{9a^2}{4},$$

from which the result follows.

5. Show that if $0 < |x - 1| < 1$, then $|4x + 1| < 9$ and $\left|\frac{1}{x + 2}\right| < 1$.

If $0 < |x - 1| < 1$, we have

$$\begin{aligned} |4x + 1| &= |4(x - 1) + 5| \\ &\le 4|x - 1| + 5 \\ &< 4 \cdot 1 + 5 \\ &\le 9. \end{aligned}$$

Also, if $0 < |x - 1| < 1$, we have

$$\begin{aligned} |x + 2| &= |(x - 1) + 3| \\ &\ge 3 - |x - 1| \\ &> 3 - 1 \\ &\ge 2 \end{aligned}$$

whence

$$\left|\frac{1}{x + 2}\right| < \frac{1}{2} < 1.$$

6. Show that if $0 < |x - 2| < 1$, then $|x + 1| < 4$ and $\left|\frac{1}{x^2 + 2x + 4}\right| < 1$.

If $|x - 2| < 1$, then

$$\begin{aligned} |x + 1| &= |(x - 2) + 3| \\ &\leq |x - 2| + 3 \\ &< 4. \end{aligned}$$

Since

$$\begin{aligned} x^2 + 2x + 4 &= \big((x - 2) + 2\big)^2 + 2\big((x - 2) + 2\big) + 4 \\ &= (x - 2)^2 + 6(x - 2) + 12 \end{aligned}$$

$$\begin{aligned} |x^2 + 2x + 4| &\geq 12 - |(x - 2)^2 + 6(x - 2)| \\ &\geq 12 - [(x - 2)^2 + 6|x - 2|]. \end{aligned}$$

Thus, if $0 < |x - 2| < 1$,

$$\begin{aligned} |x^2 + 2x + 4| &> 12 - (1 + 6) \\ &\geq 5. \end{aligned}$$

Finally, if $0 < |x - 2| < 1$, we have

$$\frac{1}{|x^2 + 2x + 4|} < \frac{1}{5} < 1.$$

7. Estimate how large $x^2 + 1$ can become if $x$ is restricted to the open interval $-3 < x < 1$.

If $-3 < x < 1$ then $3 > -x > -1$ whence

$$|x| < 3$$

and $$x^2 < 9,$$

so that $$x^2 + 1 < 10.$$

8. Use inequality properties to find a positive number $M$ such that $0 < |x - 1| < 3$ for all $x$ and

(a) $|x^2 + 2x + 4| \leq M$

(b) $|3x^2 - 2x + 3| \leq M$

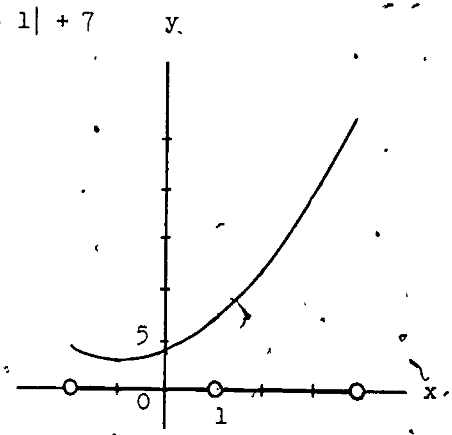
We are required to submit any positive number $M$ satisfying the given inequalities. It is not necessary to find the smallest possible number $M$.

The problem is included here to give the student preparatory experiences for Section A6-3. Because of this, the strategy is more valuable to the student than the actual solution.

(a) $|x^2 + 2x + 4| \leq M$

For $0 < |x - 1| < 3$,

$$\begin{aligned} |x^2 + 2x + 4| &= |((x - 1) + 1)^2 + 2((x - 1) + 1) + 4| \\ &= |(x - 1)^2 + 4(x - 1) + 7| \\ &\leq (x - 1)^2 + 4|x - 1| + 7 \\ &< 3^2 + 4 \cdot 3 + 7 \\ &\leq 28. \end{aligned}$$




We may take $M$ as any number, $M \geq 28$.

The graph $y = |x^2 + 2x + 4|$ shows that any number $M \geq 28$ will serve.

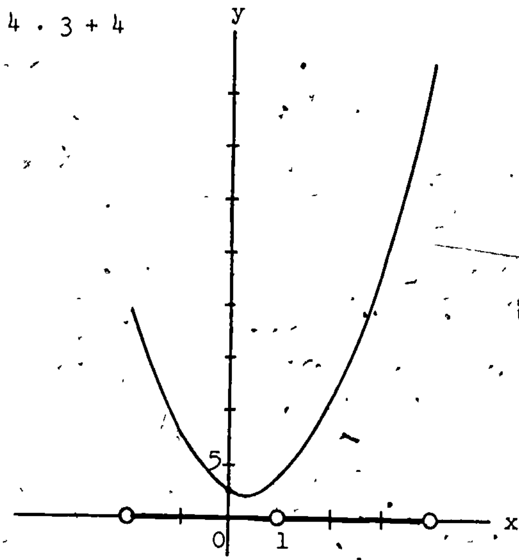
(b) $|3x^2 - 2x + 3| \leq M$

If $0 < |x - 1| < 3$, then

$$\begin{aligned} |3x^2 - 2x + 3| &= |3((x - 1) + 1)^2 - 2((x - 1) + 1) + 3| \\ &= |3(x - 1)^2 + 4(x - 1) + 4| \\ &\leq 3(x - 1)^2 + 4|x - 1| + 4 \\ &< 3 \cdot 3^2 + 4 \cdot 3 + 4 \\ &\leq 43. \end{aligned}$$

We take $M \geq 43$.

The graph of $y = |3x^2 - 2x + 3|$ shows that any number $M \geq 43$ will do.



9. (a) Show that if $0 < |x - 3| < 1$ and $0 < |x - 3| < \frac{\epsilon}{7}$, then $|x^2 - 9| < \epsilon$.

$$|x^2 - 9| = |(x - 3)((x - 3) + 6)|$$
$$\leq |x - 3| \cdot (|x - 3| + 6).$$

Thus, if $0 < |x - 3| < 1$, and $0 < |x - 3| < \frac{\epsilon}{7}$,

$$|x^2 - 9| < \frac{\epsilon}{7}(1 + 6)$$

or

$$|x^2 - 9| < \epsilon.$$

(b) Show that the pair of inequalities $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{7}$ (or $\delta \leq \min(1, \frac{\epsilon}{7})$ ) is satisfied by $\delta = \frac{\epsilon}{7 + \epsilon}$.

For $\epsilon > 0$,

$$\frac{\epsilon}{7 + \epsilon} = \frac{(7 + \epsilon) - 7}{7 + \epsilon} = \frac{7 + \epsilon}{7 + \epsilon} - \frac{7}{7 + \epsilon}$$
$$= 1 - \frac{7}{7 + \epsilon}$$
$$< 1 \quad \left(\text{since } \frac{7}{7 + \epsilon} > 0\right).$$

Also, for $\epsilon > 0$,

$$\frac{\epsilon}{7 + \epsilon} = \epsilon \cdot \frac{1}{7 + \epsilon} < \epsilon \cdot \frac{1}{7} \quad \left(\text{since } \frac{1}{7 + \epsilon} < \frac{1}{7}\right);$$

Since $\frac{\epsilon}{7 + \epsilon} < \min(1, \frac{\epsilon}{7})$, the result follows.

10. Find a number $M \geq 1$ such that $\left|\frac{x + 4}{x - 4}\right| \leq M$ for all $x$ such that $0 < |x - 2| < 1$. (See No. 3 above.)

If $0 < |x - 2| < 1$ then $\frac{1}{|x - 4|} < 1$ from Number 3 and

$$|x + 4| = |(x - 2) + 6|$$
$$\leq |x - 2| + 6$$
$$< 1 + 6 \leq 7.$$

Thus, under these conditions,

$$\left|\frac{x + 4}{x - 4}\right| < |x + 4| < 7.$$

We take $M$ as any number, $M \geq 7$.

11. For the given value of $\epsilon$, find a number $\delta$ such that if $0 < |x - 3| < \delta$, $|x^2 - 9| < \epsilon$.

(a) $\epsilon = 0.1$

(b) $\epsilon = 0.01$

Is your choice of $\delta$ in (b) acceptable as an answer in (a)? Explain.

$$\begin{aligned} |x^2 - 9| &= |x - 3| \cdot |(x - 3) + 6| \\ &\leq |x - 3| \cdot (|x - 3| + 6) \\ &< \delta(\delta + 6). \end{aligned}$$

(At the last line we used $0 < |x - 3| < \delta$.) For convenience, we restrict $\delta$ so that $\delta \leq 1$. Then, under this condition, $|x^2 - 9| < 7\delta$.

(a) To insure that $|x^2 - 9| < 0.1$ we may take $\delta = \frac{0.1}{7} = \frac{1}{70}$.

(b) To insure that $|x^2 - 9| < 0.01$ we take $\delta = \frac{0.01}{7} = \frac{1}{700}$.

The choice, $\delta = \frac{1}{700}$, is acceptable in (a), for if $0 < |x - 3| < \frac{1}{700}$ then

$$|x^2 - 9| < 7\delta \leq 0.01 < 0.1.$$

12. For the following functions, find the limit $L$ as $x$ approaches $a$. For each value of $\epsilon$, exhibit a number $\delta$ such that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$.

(a) $f(x) = 3x - 2$, $a = \frac{1}{2}$.

(b) $f(x) = mx + b$, $(m \neq 0)$.

(c) $f(x) = 1 + x^2$, $a = 0$.

(a) $\lim\limits_{x \to \frac{1}{2}} (3x - 2) = -\frac{1}{2}$.

We have

$$\begin{aligned} |(3x - 2) - (-\tfrac{1}{2})| &= |3x - \tfrac{3}{2}| \\ &= 3|x - \tfrac{1}{2}|. \end{aligned}$$

We wish to find a $\delta$ such that whenever $|x - \frac{1}{2}| < \delta$ then

$$|(3x - 2) = (-\tfrac{1}{2})| < \epsilon.$$

We take $\delta = \frac{\epsilon}{3}$. Then if $|x - \frac{1}{2}| < \delta$,

$$\begin{aligned} |(3x - \tfrac{1}{2}) - \tfrac{1}{2}| &= 3|x - \tfrac{1}{2}| \\ &< 3\delta \\ &\leq \epsilon. \end{aligned}$$

(b) $f(x) = mx + b, \ (m \neq 0)$.

$$\lim_{x \to a} f(x) = ma + b$$

$$\begin{aligned} |(mx + b) - (ma + b)| &= |m(x - a)| \\ &= |m| \cdot |x - a|. \end{aligned}$$

We wish to find a $\delta$ such that whenever $|x - a| < \delta$ then

$$|m| \cdot |x - a| < \epsilon.$$

We take $\delta = \frac{\epsilon}{|m|}$. For this choice of $\delta$, whenever $|x - a| < \delta$,

$$\begin{aligned} |(mx + b) - (ma + b)| &= |m| \cdot |x - a| \\ &< |m| \cdot \delta \\ &\leq \epsilon. \end{aligned}$$

(c) $f(x) = 1 + x^2, \quad a = 0.$

$$\lim_{x \to 0} (1 + x^2) = 1$$

$$|(1 + x^2) - 1| = x^2.$$

We wish to find a $\delta > 0$ such that $|(1 + x^2) - 1| < \epsilon$ whenever $|x - 0| < \delta$. We take $\delta = \sqrt{\epsilon}$. For this choice of $\delta$, if $|x - 0| < \delta$,

$$\begin{aligned} |(1 + x^2) - 1| &= x^2 \\ &< \delta^2 \\ &\leq \epsilon. \end{aligned}$$

TC A6-3. Epsilonic Technique

The importance of technical mastery is lost on some students, usually among the brightest. It may be necessary to emphasize for them the connection between mechanical skills and a conceptual grasp of the subject. Just as an accomplished musician can perceive the essence of a composition without stumbling over individual notes, the accomplished user of mathematics must have enough mechanical facility to be above distraction by mechanical details.

A great deal of pedagogical consideration has gone into the composition of Section A6-3. The student should develop an operationally satisfactory way of working with the idea of limit. Memorization of Definition A6-2 is certainly not sufficient. Nevertheless, definitions are like the fixed stars. They give the student a firm criterion for knowing where he is.

We wish to cultivate the attitude of inquiry in which the student asks himself the following questions:

1. Do I have suitable approximations for $L$? (The answer should be easy since the approximations are usually taken at endpoints or at interior points of a defined interval.)
2. Do I have a candidate for $L$? If so, what is it?
3. How shall I test the candidate to see if it is the limit? Can I keep the error within any given tolerance $\epsilon$ by confining the points $x$ to a suitable $\delta$-neighborhood of $a$?

It is easy to show that if, for an arbitrary $\epsilon > 0$, there exists a control, $\delta > 0$, then any smaller positive number $\delta^*$ will certainly suffice for the same $\epsilon$. For, let there exist a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. Further, let $\delta^*$ be any number, $0 < \delta^* < \delta$. It follows at once that for all $x$ such that $0 < |x - a| < \delta^*$ we have $0 < |x - a| < \delta$; whence, for all these $x$, $|f(x) - L| < \epsilon$.

We are not concerned with the largest $\delta$ that gives the desired degree of control over the error tolerance $\epsilon$; rather, we seek any number $\delta$ which is sufficient. The task is often simplified if we agree to restrict $\delta$ to numbers less than 1. This restriction simply means that we are focusing our attention on the deleted interval $\{x : 0 < |x - a| < \delta < 1\}$.

We noted in Section A6-3 (Step 1. Simplification) that frequently we are able to find a simple function $g : \delta \to c\delta$, $c$ a positive constant. This

is the case because we are dealing primarily with functions which have continuous derivatives on a full neighborhood of $a$. From the fact that $f$ is continuous, we have

$$L = \lim_{x \to a} f(x) = f(a).$$

From the fact that $f'$ is continuous on a neighborhood we know that $|f'(x)|$ is also continuous on any closed interval centered at $a$ within the neighborhood (composition of continuous functions Section 3-6). Consequently, $|f'(x)|$ has a maximum value on the interval. (Extreme Value Theorem, Section 3-7). Let $K$ be any value greater than the maximum, so that $|f'(x)| < K$ on the interval. Now, assuming $0 < |x - a| < \delta$, where this deleted $\delta$-neighborhood lies within the interval there exists a value $\xi$ within the $\delta$-neighborhood (Law of the Mean, Chapter 3) such that

$$\begin{aligned} |f(x) - L| &= |f(x) - f(a)| \\ &= |f'(\xi)(x - a)| \\ &= |f'(\xi)| \cdot |x - a| \\ &< K|x - a| \\ &< K\delta. \end{aligned}$$

The method of bounding the denominator in Example 3-3e is given because it is a routine procedure conforming to the letter of our general outline. A short cut is to anticipate the problem of bounding $x$ away from $0$ at line (1). We may recognize at once that, since $|x - a| < \delta$, the distance from $x$ to $a$ is no larger than $\delta$; we may then keep $x$ away from $0$ by requiring $\delta$ to be less than the distance $|a|$ of $a$ from $0$. To achieve this we may take $\delta \leq \frac{|a|}{2}$.

For your reference we list the following generalities:

1. The definition of limit employs only values of $x$ different from $a$.
2. Limit is a local property (sometimes called a property in the small) involving the behavior of a function within any (deleted) neighborhood of a point.

3. The existence of the limit of $f$ at a point implies that $f$ is defined for some values of $x$ in every deleted neighborhood of $a$; that is, $f(x)$ exists for some values of $x$ arbitrarily near $a$.

4. The limit is independent of the choice of the deleted neighborhood of $a$.

5. The assertion that the function $f$ has the limit $L$ as $x$ approaches $a$ is not the same as saying $f(a) = L$, nor is it the same as saying that $L$ is an upper (lower) bound of $f(x)$.

6. The value of $\delta$ depends upon the value of $\epsilon$ (exception: $f(x) = c$, $c$ constant).

A careful distinction should be made between the analysis of a problem and its exposition. This is particularly necessary in the case of limit proofs. Steps 1 and 2 show how the solution is found: in Step 1 we examine the problem and set up a simplified model; in Step 2 we outline our plan of attack or strategy. Step 3 is the actual proof where it is verified that the solution has been found. An attempt should be made to develop elegance of style in presenting proofs.

<u>Solutions Exercises A6-3</u>

In the following epsilonic arguments the analysis (Steps 1 and 2 in the pattern of the text) precedes the proof. We make liberal use of the inequalities

$$\big||a| - |b|\big| \leq |a + b| \leq |a| + |b|$$

(Section A6-1). In the selection on $\delta$ (in Numbers 4b - 4g) it is expedient to restrict $\delta$ by the auxiliary conditions that $\delta \leq 1$. The proof (verification) is simplified by an application of Exercises A6-3a, Number 5(b).

1. Prove $\lim\limits_{x \to 4} (\frac{1}{2} x - 3) = -1$: obtain an upper bound $g(\delta)$ for the absolute error and find $\delta$ in terms of $\epsilon$.

In this problem we write out the steps in detail.

To prove that $\lim\limits_{x \to 4} (\frac{1}{2} x - 3) = -1$.

For each $\epsilon > 0$ obtain a $\delta$.

Show: if $0 < |x - 4| < \delta$, then $|(\frac{1}{2} x - 3) - (-1)| < \epsilon$.

Step 1.

(a) $$|(\tfrac{1}{2} x - 3) - (-1)| = |\tfrac{1}{2} x - 2|$$
$$= \tfrac{1}{2} |x - 4|.$$

(b) If $0 < |x - 4| < \delta$,

$$\left|\left(\frac{1}{2}x - 3\right) - (-1)\right| = \frac{1}{2}|x - 4|$$
$$< \frac{1}{2}\delta.$$

(c) Take $g(\delta) = \frac{1}{2}\delta$.

Step 2. To make $g(\delta) \leq \epsilon$, set $\delta = 2\epsilon$.

Step 3. If $\delta = 2\epsilon$ and $0 < |x - 4| < \delta$, then

$$\left|\left(\frac{1}{2}x - 3\right) - (-1)\right| = \frac{1}{2}|x - 4|$$
$$< \frac{1}{2}(2\epsilon)$$
$$\leq \epsilon.$$

2. Give arguments that prove

(a) $\lim\limits_{x \to a} c = c$, $c$ any constant.

(b) $\lim\limits_{x \to a} x = a$.

(c) $\lim\limits_{x \to a} kx = ka$, $k$ any constant.

(Use the results of Example 3-3a of the text for parts b and c.)

(a) $\lim\limits_{x \to a} c = c$, $c$ constant.

Statement of problem:

For each $\epsilon > 0$ we obtain a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|c - c| < \epsilon$.

Since $|c - c| = 0$ is less than $\epsilon > 0$ for all $\delta$ we arbitrarily take any positive number for $\delta$, say $\delta = 1$.

Then for $\delta = 1$, whenever $0 < |x - a| < \delta$, we have $|c - c| < \epsilon$.

(b) $\lim\limits_{x \to a} x = a$.

From Example A6-3a we have

$$\lim_{x \to a} (mx + b) = ma + b, \quad m \neq 0,$$

whence for $m = 1$, $b = 0$,

$$\lim_{x \to a} x = a.$$

(c) $\lim_{x \to a} kx = ka$, $k$ constant.

If $k \neq 0$, the result is a direct consequence of Example 3-3a; if $k = 0$, the result follows from part (a).

3. Invoke the definition directly to prove the existence of the limits to Problem 2.

(a) See answer to Number 2(a).

(b) $\lim_{x \to a} x = a$.

We follow the pattern of Example A6-3a in Step 1. Then take $g(\delta) = \delta$.

To make $g(\delta) \leq \epsilon$ we take $\delta = \epsilon$.

Thus, if $\delta = \epsilon$ and $0 < |x - a| < \delta$, then $|x - a| < \delta \leq \epsilon$.

(c) $\lim_{x \to a} kx = ka$, $k$ constant.

We follow the pattern of Example A6-3a through Step 2 and take $\delta = \frac{\epsilon}{|m|}$ where $m = k$.

For $\delta = \frac{\epsilon}{|k|}$ and $0 < |x - a| < \delta$,

$$\begin{aligned} |kx - ka| &= |k| \cdot |x - a| \\ &< |k|\delta \\ &\leq \epsilon. \end{aligned}$$

4. In each of the following guess the limit, and then prove that your guess is correct: give an expression for $g(\delta)$ and find $\delta$ in terms of $\epsilon$.

(a) $\lim_{x \to 0} \frac{1}{1 + x^2}$

(b) $\lim_{x \to 3} \frac{x^2(x - 3)}{x - 3}$

(c) $\lim_{x \to a} \frac{x^3 - a^3}{x - a}$

(d) $\lim_{x \to 1} \frac{x + 1}{x^2 + 1}$

(e) $\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 8}$

(f) $\lim_{x \to 0} \frac{x^3 - 3x - 1}{x + 2}$

(g) $\lim_{x \to 1} \frac{4x^2 - 3x - 1}{x + 2}$

We omit repetitious material. The statement of the problem follows the pattern in the text.

(a) $\lim\limits_{x \to 0} \dfrac{1}{1 + x^2} = 1.$

$$\left|\frac{1}{1 + x^2} - 1\right| = \left|\frac{x^2}{1 + x^2}\right| = \frac{x^2}{1 + x^2}$$

$$< x^2 \qquad (\text{since } 1 + x^2 \geq 0)$$

$$< \delta^2 \qquad \text{if } 0 < |x| < \delta.$$

Take $g(\delta) = \delta^2$ and set $\delta = \sqrt{\epsilon}$.

Verification:

If $\delta = \sqrt{\epsilon}$ and $0 < |x| < \delta$, then $\left|\dfrac{1}{1 + x^2} - 1\right| < \delta^2 \leq \epsilon$.

(b) $\lim\limits_{x \to 3} \dfrac{x^2(x - 3)}{x - 3} = 9.$

$$\left|\frac{x^2(x - 3)}{x - 3} - 9\right| = |x^2 - 9| \qquad \text{for } x \neq 3$$

$$= |x - 3| \cdot |(x - 3) + 6|$$

$$\leq |x - 3| \cdot (|x - 3| + 6)$$

$$< \delta(\delta + 6).$$

(At the last line we used $0 < |x - 3| < \delta$.)

For convenience we restrict $\delta$ by requiring that $\delta \leq 1$. Under this condition if $0 < |x - 3| < \delta$

$$\left|\frac{x^2(x - 3)}{x - 3} - 9\right| < 7\delta.$$

Take $g(\delta) = 7\delta$. Obtain a $\delta$ satisfying the two conditions $\delta \leq$ and $\delta \leq \frac{\epsilon}{7}$, simultaneously. One way to do this is to take $\delta$ (Exercises A6-2, No. 9(b)).

Verification: If $\delta = \dfrac{\epsilon}{7 + \epsilon}$ and $0 < |x - 3| < \delta$

$$\left|\frac{x^2(x - 3)}{x - 3} - 9\right| < 7\delta.$$

$$\leq 7 \cdot \frac{\epsilon}{7 + \epsilon}$$

$$\leq \frac{7}{7 + \epsilon} \cdot \epsilon$$

$$< \epsilon \qquad \left(\text{since } \frac{7}{7 + \epsilon} < 1\right).$$

(c) $\lim\limits_{x\to a} \frac{x^3 - a^3}{x - a} = 3a^2$.

We have $\frac{x^3 - a^3}{x - a} = x^2 + ax + a^2$ for $x \neq a$, whence

$$\begin{aligned}\left|\frac{x^3 - a^3}{x - a} - 3a^2\right| &= |x^2 + ax - 2a^2| \\ &= \left|\left((x - a) + a\right)^2 + a\left((x - a) + a\right) - 2a^2\right| \\ &= |(x - a)^2 + 3a(x - a)| \\ &= |x - a| \cdot |(x - a) + 3a| \\ &\leq |x - a| \cdot (|x - a| + 3|a|) \\ &< \delta(\delta + 3|a|)\end{aligned}$$

(At the last line we used $0 < |x - a| < \delta$.)

For convenience we restrict $\delta$ by requiring $\delta \leq 1$. Under this condition if $0 < |x - a| < \delta$,

$$\left|\frac{x^3 - a^3}{x - a} - 3a^2\right| < \delta(1 + 3|a|).$$

We choose $\delta$ to satisfy the conditions $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{1 + 3|a|}$; i.e., $\delta \leq \min\{1, \frac{\epsilon}{1 + 3|a|}\}$. A convenient way to satisfy these conditions is to take $\delta = \frac{\epsilon}{1 + 3|a| + \epsilon}$ (Exercises A6-1, No. 6(b)).

Verification:

If $\delta = \frac{\epsilon}{1 + 3|a| + \epsilon}$ and $0 < |x - a| < \delta$, then

$$\begin{aligned}\left|\frac{x^3 - a^3}{x - a} - 3a^2\right| &= |x - a| \cdot |(x - a) + 3a| \\ &\leq |x - a| \cdot (|x - a| + 3|a|) \\ &< \delta(1 + 3|a|) \\ &\leq \frac{\epsilon}{1 + 3|a| + \epsilon} \cdot (1 + 3|a|) \\ &\leq \epsilon \cdot \frac{1 + 3|a|}{1 + 3|a| + \epsilon} \\ &< \epsilon \qquad \left(\text{since } \frac{1 + 3|a|}{1 + 3|a| + \epsilon} < 1\right).\end{aligned}$$

(d) $\lim\limits_{x \to 1} \frac{x+1}{x^2+1} = 1.$

$$\left|\frac{x+1}{x^2+1} - 1\right| = \left|\frac{x - x^2}{x^2+1}\right| = \frac{|x-1| \cdot |x|}{1+x^2}$$

$$\leq |x-1| \cdot |x| \qquad (\text{since } 1 + x^2 \geq 1)$$

$$\leq |x-1| \cdot |(x-1) + 1|$$

$$\leq |x-1| \cdot (|x-1| + 1)$$

$$< \delta(\delta + 1) \qquad \text{if } |x-1| < \delta.$$

For convenience we restrict $\delta$ by requiring $\delta \leq 1$. Under this condition if $|x - 1| < \delta$

$$\left|\frac{x+1}{x^2+1} - 1\right| < \delta(\delta + 1)$$

$$\leq 2\delta.$$

To satisfy the two conditions $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{2}$ simultaneously we take (for convenience) $\delta = \frac{\epsilon}{2+\epsilon}$.

Verification:
If $\delta = \frac{\epsilon}{2+\epsilon}$ and $|x - 1| < \delta$,

$$\left|\frac{x+1}{x^2+1} - 1\right| < 2\delta$$

$$\leq 2 \cdot \frac{\epsilon}{2+\epsilon}$$

$$\leq \frac{2}{2+\epsilon} \cdot \epsilon$$

$$< \epsilon \qquad (\text{since } \frac{2}{2+\epsilon} < 1).$$

(e) $\lim\limits_{x \to 2} \frac{x^2-4}{x^3-8} = \frac{1}{3}.$

$$\left|\frac{x^2-4}{x^3-8} - \frac{1}{3}\right| = \left|\frac{x+2}{x^2+2x+4} - \frac{1}{3}\right| \qquad (x \neq 2)$$

$$= \left|\frac{-x^2+x+2}{3(x^2+2x+4)}\right|$$

$$= \frac{1}{3}\left|\frac{(x-2)(x+1)}{x^2+2x+4}\right|.$$

For convenience we restrict $\delta$ by requiring $\delta \leq 1$. Under this condition, if $0 < |x - 2| < \delta$,

$$\left|\frac{x^2 - 4}{x^3 - 8} - \frac{1}{3}\right| < \frac{1}{3}\,|(x - 2)(x + 1)| \quad \text{(from Exercises A6-2, No. 6)}$$

$$\leq \frac{1}{3}\,|x - 2| \cdot |(x - 2) + 3|$$

$$\leq \frac{1}{3}\,|x - 2| \cdot (|x - 2| + 3)$$

$$< \frac{1}{3}\,\delta(1 + 3)$$

$$\leq \frac{4}{3}\,\delta.$$

We wish to obtain a value $\delta$ satisfying the two conditions $\delta \leq 1$ and $\delta \leq \frac{3}{4}\,\epsilon$, simultaneously. One way to satisfy the conditions is to set

$$\delta = \frac{\frac{3}{4}\,\epsilon}{1 + \frac{3}{4}\,\epsilon} = \frac{3\epsilon}{4 + 3\epsilon} \quad \text{(Exercises A6-1, No. 6(b))}.$$

Verification:

If $\delta = \frac{3\epsilon}{4 + 3\epsilon}$ and $0 < |x - 2| < \delta$,

$$\left|\frac{x^2 - 4}{x^3 - 8} - \frac{1}{3}\right| \leq \frac{1}{3}\,|x - 2| \cdot (|x - 2| + 3)$$

$$< \frac{4}{3}\,\delta$$

$$\leq \frac{4}{3} \cdot \frac{3\epsilon}{4 + 3\epsilon}$$

$$\leq \frac{12}{3(4 + 3\epsilon)} \cdot \epsilon$$

$$< \epsilon \qquad \left(\text{since } \frac{12}{3(4 + 3\epsilon)} < 1\right).$$

(f) $\lim\limits_{x \to 0} \dfrac{x^3 - 3x - 1}{x + 2} = -\dfrac{1}{2}$.

$$\left|\frac{x^3 - 3x - 1}{x + 2} - \left(-\frac{1}{2}\right)\right| = \left|\frac{2x^3 - 5x}{2(x + 2)}\right|.$$

For convenience we restrict $\delta$ by requiring $\delta \leq 1$. Thus if $0 < |x| < \delta \leq 1$, $|x + 2| \geq 2 - |x| \geq 1$ and $\frac{1}{|x + 2|} \leq 1$.

Thus if $\delta \leq 1$ and $0 < |x| < \delta$, we have

$$\left|\frac{x^3 - 3x - 1}{x + 2} - (-\frac{1}{2})\right| = \left|\frac{2x^3 - 5x}{2(x + 2)}\right|$$

$$\leq \frac{1}{2} |2x^3 - 5x|$$

$$\leq \frac{1}{2} \cdot |x| \cdot |2x^2 - 5|$$

$$\leq |x| \cdot (x^2 + \frac{5}{2})$$

$$< \delta \cdot (1 + \frac{5}{2})$$

$$< 4\delta \qquad \text{(since } \frac{7}{2}\delta < 4\delta\text{)}.$$

To satisfy the two conditions $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{4}$ simultaneously, we take, for convenience, $\delta = \frac{\epsilon}{4 + \epsilon}$.

Verification:

Set $\delta \leq \frac{\epsilon}{4 + \epsilon}$ in the statement of the problem, as in (a) - (e). In the last step we have, under the conditions that $0 < |x - 0| < \delta$,

$$\left|\frac{x^3 - 3x - 1}{x + 2} - (-\frac{1}{2})\right| < 4\delta$$

$$\leq 4 \cdot \frac{\epsilon}{4 + \epsilon}$$

$$\leq \epsilon \qquad \text{(since } \frac{4}{4 + \epsilon} \leq 1\text{)}.$$

(g) $\lim_{x \to 1} \frac{4x^2 - 3x - 1}{x + 2} = 0.$

$$\left|\frac{4x^2 - 3x - 1}{x + 2}\right| = \frac{|(x - 1)(4x + 1)|}{|x + 2|}$$

$$= \frac{|x - 1| \cdot |4(x - 1) + 5|}{|x + 2|}$$

We restrict $\delta$ so that $\delta \leq 1$. Under this restriction, if $0 < |x - 1| < \delta$, we have

$$\left|\frac{4x^2 - 3x - 1}{x + 2}\right| < |x - 1| \cdot |4(x - 1) + 5| \quad \text{(Exercises A6-2, No. 5)}$$

$$\leq |x - 1| \cdot (4|x - 1| + 5)$$

$$< \delta(4\delta + 5)$$

$$\leq 9\delta.$$

We require that $\delta \le \min(1, \frac{\epsilon}{9})$. This condition is satisfied if we take $\delta = \frac{\epsilon}{9+\epsilon}$.

Verification:

If $\delta = \frac{\epsilon}{9+\epsilon}$ and $0 < |x - 1| < \delta$, then

$$\begin{aligned}
\left|\frac{4x^2 - 3x - 1}{x + 2}\right| &< |(x - 1)(4x + 1)| \\
&\le |x - 1| \cdot (4|x - 1| + 5) \\
&< 9\delta \\
&\le 9 \cdot \frac{\epsilon}{9+\epsilon} \\
&< \epsilon \qquad \text{(since } \tfrac{\epsilon}{9+\epsilon} < 1\text{)}.
\end{aligned}$$

TC A6-4. Limit Theorems

Theorems become tools to the student only in so far as he understands the hypotheses and appreciates conclusions that derive therefrom. Graphical considerations are usually helpful in visualizing abstract properties and proofs (in particular, Theorems A6-4a, b, c, f). Proofs of the theorems should be made plausible, but memorization of formal proofs is inconsistant with the philosophy of this course.

The statement $\lim_{x \sim a} f(x) = L$ simply means that $L$ satisfies the conditions of Definition A6-2. It is conceivable that another number $M$ might satisfy the same conditions. Then the statements $\lim_{x \sim a} f(x) = L$ and $\lim_{x \sim a} f(x) = M$ would both be true. If this could happen, then $\lim_{x \sim a} f(x)$ could have no meaning by itself. Consequently, each limit theorem would need to be appropriately interpreted; for example, Theorem A6-4c would be stated:

> THEOREM A6-4c. If $\lim_{x \sim a} f(x) = L$ and $\lim_{x \sim a} g(x) = M$, then $L + M$ is a limit of $g : g$ at $x = a$.

This interpretation is unnecessary, since $\lim_{x \sim a} f(x)$, if it exists, is unique.

> THEOREM. If $L$ and $M$ are limits of $f$ at $a$, that is, if $\lim_{x \sim a} f(x) = L$ and $\lim_{x \sim a} f(x) = M$, then $L = M$.

Proof. Suppose $L > M$, take $\epsilon = \frac{L - M}{3} > 0$ then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$-\epsilon < L - f(x) < \epsilon \quad \text{for } 0 < |x - a| < \delta_1,$$

$$-\epsilon < f(x) - M < \epsilon \quad \text{for } 0 < |x - a| < \delta_2.$$

Hence both inequalities hold for

$$0 < |x - a| < \min\{\delta_1, \delta_2\}.$$

Thus $0 < L - M < 2\epsilon$. Impossible

Sometimes assumptions are left tacit. For example, in the proof of Theorem A6-4c, $\epsilon^*$ is, of course, subject to the conditions imposed upon $\epsilon$, that is, $\epsilon^* > 0$.

In Theorem A6-4c, there is a tolerance, $\epsilon$, for the sum $(f + g)$ and tolerances, $\epsilon_1$ and $\epsilon_2$, for the addends ( $f$ and $g$, respectively). It is sufficient that these tolerances satisfy the condition $\epsilon_1 + \epsilon_2 \le \epsilon$: for convenience (and for definiteness) we take $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2}$, in the text. The control $\delta_1$ maintains the tolerance $\epsilon_1 = \frac{\epsilon}{2}$ for $f$ and control $\delta_2$ maintains $\epsilon_2 = \frac{\epsilon}{2}$ for $g$. It follows that $\delta \le \min\{\delta_1, \delta_2\}$ will maintain the tolerance $\frac{\epsilon}{2}$ for $f$ and also for $g$.

We can obtain a slightly different proof of Theorem A6-4d by utilizing Lemma A6-4 in the following way. By hypothesis, corresponding to any positive $\epsilon_1$, $\epsilon_2$ we can find $\delta_1$, $\delta_2$ such that

$$|f(x) - L| < \epsilon_1 \text{ whenever } 0 < |x - a| < \delta_1,$$

$$|g(x) - M| < \epsilon_2 \text{ whenever } 0 < |x - a| < \delta_2,$$

and by Lemma A6-4 there is a $\delta^*$ such that

$$|g(x)| < \frac{3}{2}|M| \text{ whenever } 0 < |x - a| < \delta^*.$$

Since $f(x)g(x) - LM = (f(x) - L)g(x) + L(g(x) - M)$, if $|f(x) - L| < \epsilon_1$, $|g(x) - M| < \epsilon_2$, and $|g(x)| < \frac{3}{2}|M|$ then

$$|f(x)g(x) - LM| \le |f(x) - L| \cdot |g(x)| + |L| \cdot |g(x) - M|$$

$$< \epsilon_1 \frac{3}{2}|M| + |L|\epsilon_2.$$

In order to remain within the tolerance $\epsilon$ we take $\epsilon_1 = \epsilon_2$ and

$$\epsilon_1 \frac{3}{2}|M| + |L|\epsilon_2 = \epsilon_1(\frac{3}{2}|M| + |L|) \le \epsilon.$$

Then to cover the case where $M = L = 0$ we take

$$\epsilon_1 = \epsilon_2 = \frac{\epsilon}{\frac{3}{2}|M| + |L| + 1}.$$

Let $\delta_1$, $\delta_2$, and $\delta^*$ be appropriate controls for this choice of $\epsilon_1$ and $\epsilon_2$ and take

$$\delta = \min\{\delta_1, \delta_2, \delta^*\}.$$

For this choice of $\delta$ and for

$$0 < |x - a| < \delta$$

we have

$$|f(x)g(x) - LM| \leq |f(x) - L| \cdot |g(x)| + |L| \cdot |g(x) - M|$$

$$\leq \frac{\epsilon}{\frac{3}{2}|M| + |L| + 1} \cdot \left(\frac{3}{2}|M| + |L|\right)$$

$$< \epsilon.$$

Sum notation may be used to express a linear combination

$$\phi(x) = \sum_{i=1}^{n} c_i f_i(x)$$

and state the corollary to Theorem A6-4c,

$$\lim_{x \to a} \sum_{i=1}^{n} c_i f_i(x) = \sum_{i=1}^{n} c_i \lim_{x \to a} f_i(x)$$

$$= \sum_{i=1}^{n} c_i L_i.$$

Mathematical induction (Appendix 3) is required for the proof of the corollaries to Theorems A6-4c and A6-4d, respectively. If the student works through these proofs he will gain a deeper understanding of the fundamental results and an appreciation of the power of mathematical induction.

In Lemma A6-4 we have "... a $\delta$-neighborhood of $a$ wherein $g(x)$ is closer to $M$ than to zero." This means: for all $x$ in the $\delta$-neighborhood and in the domain of $g$, $g(x)$ is closer to $M$ than to zero.

SQUEEZE THEOREM (Proof). Let $I$ be the deleted neighborhood of $a$ where

$$h(x) < f(x) \leq g(x).$$

For every $\epsilon > 0$, $I$ contains a deleted $\delta$-neighborhood of $a$ wherein $|h(x) - M| < \epsilon$ and $|g(x) - M| < \epsilon$. Equivalently, we have

$$M - \epsilon < h(x), \quad g(x) < M + \epsilon$$

whenever $0 < |x - a| < \delta$. It follows that

$$M - \epsilon < f(x) < M + \epsilon$$

or

$$|f(x) - M| < \epsilon$$

whenever $0 < |x - a| < \delta$.

## Solutions Exercises A6-4

1. Prove the corollary to Theorem A6-4c. The limit of a linear combination of functions is the same linear combination of the limits of the functions; i.e., if $\lim_{x \to a} f_i(x) = L_i$, $i = 1, 2, \ldots, n$, then

$$\lim_{x \to a} \sum_{i=1}^{n} c_i f_i(x) = \sum_{i=1}^{n} \lim_{x \to a} c_i f_i(x)$$

$$= \sum_{i=1}^{n} c_i \lim_{x \to a} f_i(x)$$

$$= \sum_{i=1}^{n} c_i L_i \qquad \text{(see A6-2)}.$$

Proof. We use the First Principle of Mathematical Induction (Appendix 3-1), and take for $A_n$ the assertion

$$\lim_{x \to a} \sum_{i=1}^{n} c_i f_i(x) = \sum_{i=1}^{n} c_i L_i .$$

For $n = 1$ we have as assertion $A_1$,

$$\lim_{x \to a} c_1 f_1(x) = c_1 L_1 ,$$

which is true by Theorem A6-4b.

We now assume $A_n$ true for $n = k$ and seek to prove that $A_{k+1}$ is true. From the induction hypothesis,

$$\lim_{x \sim a} \sum_{i=1}^{k} c_i f_i(x) = \sum_{i=1}^{k} c_i L_i .$$

Now

$$\lim_{x \sim a} \sum_{i=1}^{k+1} c_i f_i(x) = \lim_{x \sim a} \left( \sum_{i=1}^{k} c_i f_i(x) + c_{k+1} f_{k+1}(x) \right)$$

$$= \lim_{x \sim a} \sum_{i=1}^{k} c_i f_i(x) + \lim_{x \sim a} c_{k+1} f_{k+1}(x) \text{ (Theorem A6-4c)}$$

$$= \sum_{i=1}^{k} c_i L_i + \lim_{x \sim a} c_{k+1} f_{k+1}(x)$$

$$= \sum_{i=1}^{k} c_i L_i + c_{k+1} L_{k+1} \qquad \text{(Theorem A6-4b)}$$

$$= \sum_{i=1}^{k+1} c_i L_i ,$$

and assertion $A_{k+1}$ is true. Therefore,

$$\lim_{x \sim a} \sum_{i=1}^{n} c_i f_i(x) = \sum_{i=1}^{n} c_i L_i$$

holds for every natural number $n$.

2. Prove the corollary to Theorem A6-4d. For any polynomial function $p$,

$$\lim_{x \sim a} p(x) = p(a)$$

Proof. To establish the corollary, we use the First Principle of Mathematical Induction to prove first that

$$\lim_{x \sim a} x^n = a^n, \qquad n \text{ any natural number.}$$

We take for $A_n$ the assertion that $\lim_{x \sim a} x^n = a^n$.

For $n = 1$, assertion $A_1$ is

$$\lim_{x \to a} x = a$$

which is true (Exercises A6-3, No. 2(b)).

We now assume $A_k$ true and try to prove $A_{k+1}$ true. The induction hypothesis is

$$\lim_{x \to a} x^k = a^k.$$

Now

$$\begin{aligned}
\lim_{x \to a} x^{k+1} &= \lim_{x \to a} (x^k \cdot x) \\
&= \left(\lim_{x \to a} x^k\right)\left(\lim_{x \to a} x\right) && \text{(Theorem A6-4d)} \\
&= a^k \cdot a && (A_1 \text{ and induction hypothesis}) \\
&= a^{k+1}.
\end{aligned}$$

Thus, $A_{k+1}$ is true and

$$\lim_{x \to a} x^n = a^n, \quad n \text{ any natural number.}$$

To complete the proof, let

$$p(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n = \sum_{i=0}^{n} c_i x^i.$$

$$\begin{aligned}
\lim_{x \to a} p(x) &= \lim_{x \to a} \sum_{i=0}^{n} c_i x^i \\
&= \sum_{i=0}^{n} \lim_{x \to a} c_i x^i && \text{(Corollary to Theorem A6-4c)} \\
&= \sum_{i=0}^{n} c_i \lim_{x \to a} x^i && \text{(Theorem A6-4d)} \\
&= \sum_{i=0}^{n} c_i a^i \\
&= p(a),
\end{aligned}$$

as was to be shown.

3. Prove the corollaries to Lemma A6-4.

(a) Corollary 1. If $\lim_{x \to a} g(x) = M$ and $M \neq 0$, then there exists a neighborhood of $a$ where $|\frac{3M}{2}| > |g(x)| > |\frac{M}{2}|$ for $x$ in the domain of $g$.

Proof. Since $g$ has limit $M$ at $a$, there is a δ-neighborhood of $a$ in which $g(x)$ is closer to $M$ than to zero. We consider two cases: $M > 0$ and $M < 0$.

Case 1. If $M > 0$, we have $g(x) > 0$ by Lemma A6-4, and $\frac{3M}{2} > g(x) > \frac{M}{2} > 0$ or $|\frac{3M}{2}| > |g(x)| > |\frac{M}{2}| > 0$.

Case 2. If $M < 0$, we have $-M > 0$ and $\lim_{x \to a} (-g(x)) = -M > 0$. Thus, $-g(x) > 0$ by Lemma A6-4, and $\frac{-3M}{2} > -g(x) > \frac{-M}{2} > 0$. This inequality is equivalent to $|\frac{-3M}{2}| > |-g(x)| > |\frac{-M}{2}| > 0$ or $|\frac{3M}{2}| > |g(x)| > |\frac{M}{2}| > 0$, which is the same as the inequality obtained in Case 1.

(b) Corollary 2. A limit of a function whose values are nonnegative is nonnegative.

Proof. Let $g(x) \geq 0$ and let $\lim_{x \to a} g(x) = M$. We want to show that $M \geq 0$. To do this we show that the assumption $M < 0$ leads to a contradiction.

If $M < 0$, then $-M > 0$ and

$$\lim_{x \to a} \left(-g(x)\right) = -M > 0.$$

Therefore,

$$-g(x) > 0 \qquad \text{(Lemma A6-4)}$$

or

$$g(x) < 0$$

contradicting the hypothesis that $g(x) \geq 0$. Thus, $M \geq 0$.

4. Prove the corollaries to Theorem A6-4e

(a) Corollary 1. If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$ where $M \neq 0$,

then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$.

Proof. Since $M \neq 0$, by Theorem 3-4e, we have

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}.$$

From Theorem A6-4d, we have

$$\begin{aligned}
\lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \left(f(x) \cdot \frac{1}{g(x)}\right) \\
&= \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} \frac{1}{g(x)}\right) \\
&= L \cdot \frac{1}{M} \\
&= \frac{L}{M}.
\end{aligned}$$

(b) Corollary 2. If $p$ and $q$ are polynomials, and if $q(a) \neq 0$, then

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

Proof. By the corollary to Theorem A6-4d,

$$\lim_{x \to a} p(x) = p(a) \quad \text{and} \quad \lim_{x \to a} q(x) = q(a).$$

Since $q(a) \neq 0$,

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ (Corollary 1 to Theorem A6-4e).}$$

5. Find the following limits, giving at each step the theorem on limits which justifies it:

(a) $\lim\limits_{x \to 3} (2 + x) = \lim\limits_{x \to 3} 2 + \lim\limits_{x \to 3} x$ Theorem A6-4c

$= 2 + 3$ Theorem A6-4a, Example A6-3a

$= 5.$

(b) $\lim_{x \to -1} (5x - 2) = \lim_{x \to -1} 5x + \lim_{x \to -1} (-2)$ — Theorem A6-4c

$= [illegible] + (-2)$ — Theorem A6-4b, Example A6-3a, Theorem A6-4a

$= -7.$

(c) $\lim_{x \to 0} \left(\frac{a}{1 + |x|} - b\sqrt{|x|}\right)$, where $a$ and $b$ are constants.

$\lim_{x \to 0} \left(\frac{a}{1 + |x|} - b\sqrt{|x|}\right) = \lim_{x \to 0} \frac{a}{1 + |x|} + \lim_{x \to 0} (-b\sqrt{|x|})$ — Theorem A6-4c

$= a \lim_{x \to 0} \frac{1}{1 + |x|} - b \lim_{x \to 0} \sqrt{|x|}$ — Theorem A6-4b

$= a \cdot 1 - b \cdot 0$ — Example A6-3b, Theorem A6-4e, Example A6-3c

$= a.$

(d) $\lim_{x \to a} (x^3 + ax^2 + a^2x + a^3)$, where $a$ is constant.

$\lim_{x \to a} (x^3 + ax^2 + a^2x + a^3) = a^3 + a \cdot a^2 + a^2 \cdot a + a^3$ — Corollary to Theorem A6-4d

$= 4a^3.$

6. Find the following limits, giving at each step the theorem which justifies it.

(a) $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)}$

$= \lim_{x \to 1} \left(\frac{x - 1}{x - 1}\right) \cdot \lim_{x \to 1} \left(\frac{x^2 + x + 1}{x + 1}\right)$ — Theorem A6-4d

$= 1 \cdot \frac{3}{2}$ — Corollary 2 to Theorem A6-4e, Corollary to Theorem A6-4d

$= \frac{3}{2}.$

(b) $\lim\limits_{x\to 3} \frac{x^2 - 9}{x^3 - 27} = \lim\limits_{x\to 3} \frac{(x - 3)(x + 3)}{(x - 3)(x^2 + 3x + 9)}$

$= \lim\limits_{x\to 3} \left(\frac{x - 3}{x - 3}\right) \cdot \lim\limits_{x\to 3}\left(\frac{x + 3}{x^2 + 3x + 9}\right)$ Theorem A6-4d

$= 1 \cdot \frac{6}{27}$ Corollary 2 to Theorem A6-4e; Corollary to Theorem A6-4d

$= \frac{2}{9}$

7. Find $\lim\limits_{x\to 1} \frac{x^n - 1}{x - 1}$, for $n$ a positive integer. Verify first that

$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \ldots + x + 1 \quad (x \neq 1).$

The verification of the division is done by simple algebraic methods.

Now let $p(x) = x^{n-1} + x^{n-2} + \ldots + x + 1.$

$\lim\limits_{x\to 1} p(x) = p(1)$ Corollary to Theorem A6-4d

$= n.$

8. Determine whether the following limits exist and, if they do exist, find their values.

(a) $\lim\limits_{x\to 1} \frac{1 + \sqrt{x}}{1 - x}$ does not exist.

$\lim\limits_{x\to 1} \frac{1 + \sqrt{x}}{1 - x} = \lim\limits_{x\to 1} \frac{1}{1 - \sqrt{x}}$ which does not exist.

(b) $\lim\limits_{x\to a} (x^n - a^n)$, $n$ a positive integer, $a$ constant.

Let $p(x) = x^n - a^n$.

$\lim\limits_{x\to a} p(x) = p(a)$

$= a - a$

$= 0.$

(c) $\lim_{x\to -1} \frac{\sqrt{2+x}+1}{x+1}$

$\lim_{x\to -1} (\sqrt{2+x}+1) = 2$ and $\lim_{x\to -1}(x+1) = 0$, hence $\lim_{x\to -1} \frac{\sqrt{2+x}+1}{x+1}$ does not exist.

(d) $$\lim_{x\to 1} \frac{(x-2)(\sqrt{x}-1)}{x^2+x-2} = \lim_{x\to 1} \frac{(x-2)(\sqrt{x}-1)}{(x+2)(x-1)}$$

$$= \lim_{x\to 1} \frac{(x-2)(\sqrt{x}-1)}{(x+2)(\sqrt{x}+1)(\sqrt{x}-1)}$$

$$= \lim_{x\to 1} \frac{x-2}{(x+2)(\sqrt{x}+1)} = -\frac{1}{6}.$$

(e) $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}.$

9. Using the algebra of limits show that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = L$ if and only if $\lim_{x\to a} \frac{f(x)-f(a)-L(x-a)}{|x-a|} = 0$.

First, we note that $\lim_{x\to a} g(x) = 0$ if and only if $\lim_{x\to a} |g(x)| = 0$, and also that $|g(x)| = \left||g(x)|\right|$.

Let

$$A = \frac{f(x)-f(a)}{x-a} - L,$$

and

$$B = \frac{f(x)-f(a)-L(x-a)}{|x-a|}.$$

Then,

$$|A| = |B|.$$

Part 1. Assume $\lim_{x\to a} B = 0$. This implies that $\lim_{x\to a} |A| = 0$ and, thus,

$$\lim_{x\to a} A = 0 \text{ or } \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = L.$$

Part 2. Assume $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = L$. This implies that $\lim_{x\to a} A = 0$ and, thus

$$\lim_{x\to a} |B| = 0 \text{ and } \lim_{x\to a} B = 0.$$

10. Assume $\lim_{x \to 0} \sin x = 0$ and $\lim_{x \to 0} \cos x = 1$. Find each of the following limits, if the limit exists, giving at each step the theorem on limits which justifies it.

(a) $\lim_{x \to 0} \sin^3 x = \left(\lim_{x \to 0} \sin^2 x\right) \cdot \left(\lim_{x \to 0} \sin x\right)$ Theorem A6-4d

$= \left(\lim_{x \to 0} \sin x \cdot \lim_{x \to 0} \sin x\right) \cdot \left(\lim_{x \to 0} \sin x\right)$ Theorem A6-4d

$= 0.$

(b) $\lim_{x \to 0} \tan x = \lim_{x \to 0} \frac{\sin x}{\cos x}$

$= \frac{\lim_{x \to 0} \sin x}{\lim_{x \to 0} \cos x} = 0.$ Corollary 1 to Theorem A6-4e

(c) $\lim_{x \to 0} \sin 2x = \lim_{x \to 0} (2 \sin x \cos x)$

$= 2 \cdot \left(\lim_{x \to 0} \sin x \cos x\right)$ Theorem A6-4b

$= 2 \cdot \left(\lim_{x \to 0} \sin x\right) \cdot \left(\lim_{x \to 0} \cos x\right)$ Theorem A6-4d

$= 0.$

(d) $\lim_{x \to 0} \frac{\sin x}{\tan x} = \lim_{x \to 0} \frac{\sin x \cdot \cos x}{\sin x}$

$= \lim_{x \to 0} \frac{\sin x}{\sin x} \cdot \lim_{x \to 0} \cos x$ Theorem A6-4d

$= 1.$

(e) $\lim_{x \to 0} \frac{1 - \cos x}{\sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x} \cdot \frac{1 + \cos x}{1 + \cos x}$ Theorem A6-4d

$= \lim_{x \to 0} \frac{\sin^2 x}{\sin x} \cdot \lim_{x \to 0} \frac{1}{1 + \cos x}$ Theorem A6-4d

$= 0 \cdot \frac{1}{2} = 0.$

(f) $\lim_{x \to 0} \frac{\cos 2x}{\cos x + \sin x} = \lim_{x \to 0} \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x}$

$= \lim_{x \to 0} (\cos x - \sin x) \cdot \lim_{x \to 0} \frac{\cos x + \sin x}{\cos x + \sin x}$ Theorem A6-4d

$= \left[\lim_{x \to 0} \cos x - \lim_{x \to 0} \sin x\right] \cdot 1$ Theorem A6-4c

$= 1.$

11. Prove the corollaries to Theorem A6-4f.

(a) Corollary 1 (Sandwich Theorem). If $h(x) \le f(x) \le g(x)$ in some deleted neighborhood of $a$, and if $\lim_{x \to a} h(x) = K$ and $\lim_{x \to a} g(x) = M$, then, if $\lim_{x \to a} f(x)$ exists, $K \le \lim_{x \to a} f(x) \le M$.

Pooof. Since $f(x) \le g(x)$ in a deleted neighborhood of $a$, we have, by Theorem A6-4f, that

$$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x).$$

Again, since $h(x) \le f(x)$ in a deleted neighborhood of $a$, we have, by Theorem A6-4f, that

$$\lim_{x \to a} h(x) \le \lim_{x \to a} f(x).$$

Thus,

$$\lim_{x \to a} h(x) \le \lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$$

or

$$K \le \lim_{x \to a} f(x) \le M.$$

(b) Corollary 2 (Squeeze Theorem). (Hint: Prove $\lim_{x \to a} f(x)$ exists.)

Because the proof of the Squeeze Theorem does not follow immediately from Theorem A6-4f it is given in Section TCA6-4 immediately preceding Solutions Exercises A6-4.

12. For what integral values of $m$ and $n$ does $\lim\limits_{x \to -a} \frac{x^m + a^m}{x^n + a^n}$ exist? Find the limit for these cases.

Part 1. For $a = 0$:

(i) if $m > n$,

$$\lim_{x \to 0} \frac{x^m}{x^n} = \lim_{x \to 0} x^{m-n} = 0;$$

(ii) if $m = n$,

$$\lim_{x \to 0} \frac{x^m}{x^n} = \lim_{x \to 0} 1 = 1;$$

(iii) if $m < n$, $\lim\limits_{x \to 0} \frac{x^m}{x^n}$ does not exist.

Part 2. For $a \neq 0$:

$$\lim_{x \to -a} (x^p + a^p) = \begin{cases} 2a^p, & p \text{ even} \\ 0, & p \text{ odd.} \end{cases}$$

(i) Thus, for $a \neq 0$ and $n$ even,

$$\lim_{x \to -a} \frac{x^m + a^m}{x^n + a^n} = \frac{\lim\limits_{x \to -a} (x^m + a^m)}{\lim\limits_{x \to -a} (x^n + a^n)}$$

$$= \begin{cases} \dfrac{2a^m}{2a^n} = a^{m-n}, & m \text{ even} \\[2ex] \dfrac{0}{2a^n} = 0, & m \text{ odd.} \end{cases}$$

(ii) For $a \neq 0$, $n$ odd, $m$ even, $\lim\limits_{x \to -a} \frac{x^m + a^m}{x^n + a^n}$ does not exist.

(iii) For $a \neq 0$, $n$ odd, $m$ odd,

$$\lim_{x \to -a} \frac{x^m + a^m}{x^n + a^n} = \lim_{x \to -a} \frac{(x + a)(x^{m-1} - ax^{m-2} + \ldots + a^{m-1})}{(x + a)(x^{n-1} - ax^{n-2} + \ldots + a^{n-1})}$$

$$= \frac{ma^{m-1}}{na^{n-1}}$$

$$= \frac{m}{n} a^{m-n}$$

13. Prove that if $\lim_{x \to a} f(x) = 0$ and $g(x)$ is bounded in a neighborhood of $x = a$, then $\lim_{x \to a} f(x) \cdot g(x) = 0$.

<u>Proof</u>. Since $g(x)$ is bounded there exists a positive number $M$ such that

$$-M < g(x) < M$$

in a neighborhood of $x = a$. Consequently,

$$-M \cdot |f(x)| \leq f(x)g(x) \leq M \cdot |f(x)|$$

in a neighborhood of $x = a$.

By the Squeeze Theorem,

$$\lim_{x \to a} f(x) \cdot g(x) = 0.$$

14. (a) Verify that if $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and if $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} f(x) = 0$.

$$\lim_{x \to a} \frac{f(x)}{g(x)} \cdot \lim_{x \to a} g(x) = \lim_{x \to a} \left( \frac{f(x)}{g(x)} \cdot g(x) \right)$$

or

$$0 = \lim_{x \to a} f(x). \qquad \text{Theorem A6-4d}$$

(b) Give examples of functions $f$ and $g$ for which $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ yet the limit of their quotient does not exist.

If $f(x) = x^m$ and $g(x) = x^n$, $m < n$, $\frac{f(x)}{g(x)}$ does not have a limit at 0. Of course, many other examples exist.

15. Prove that if $\lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} f(x)$ does not exist, then the limit of the quotient $\frac{f(x)}{g(x)}$ does not exist.

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ does exist and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} f(x) = 0$ by by Number 14(a). Contradiction.

16. The right-hand limit at a point $P(p, f(p))$ of a function is the limit of the function at the point $P$ for a right-hand domain $(p, p + \delta)$. Similarly, for the left-hand limit, the domain is restricted to $(p - \delta, p)$. We denote them symbolically by $\lim_{x \to p^+} f(x)$ and $\lim_{x \to p^-} f(x)$, respectively.

In particular, $\lim_{x \to 2^+} [x] = 2$, $\lim_{x \to 2^-} [x] = 1$. Determine the indicated limits, if they exist, of the following:

(a) $\lim_{x \to 2^+} \frac{[x]^2 - 4}{x^2 - 4}$.

For $x \in (2, 2 + \delta)$, $0 < \delta < 1$,

$$[x]^2 - 4 = ([x] - 2)([x] + 2)$$
$$= 0 \cdot 4 = 0.$$

Thus

$$\lim_{x \to 2^+} \frac{[x]^2 - 4}{x^2 - 4} = \lim_{x \to 2^+} 0 = 0.$$

(b) $\lim_{x \to 2^-} \frac{[x]^2 - 4}{x^2 - 4}$.

For $x \in (2 - \delta, 2)$, $0 < \delta < 1$,

$$[x]^2 - 4 = (-1)(3) = -3.$$

Thus

$$\lim_{x \to 2^-} \frac{[x]^2 - 4}{x^2 - 4} = \lim_{x \to 2^-} \frac{3}{x^2 - 4}$$

does not exist.

(c) $\lim\limits_{x \to 3^+} (x - 2 + [2 - x] - [x])$.

For $x \in (3, 3 + \delta)$, $0 < \delta < 1$,

$$[2 - x] = -2 \text{ and } [x] = 3.$$

Thus

$$\lim_{x \to 3^+}(x - 2 + [2 - x] - [x]) = \lim_{x \to 3^+}(x - 2 - 2 - 3)$$
$$= -4.$$

(d) $\lim\limits_{x \to 3^-}(x - 2 + [2 - x] - [x])$.

For $x \in (3 - \delta, 3)$, $0 < \delta < 1$,

$$[2 - x] = -1 \text{ and } [x] = 2.$$

Thus,

$$\lim_{x \to 3^-}(x - 2 + [2 - x] - [x]) = \lim_{x \to 3^-}(x - 2 - 1 - 2)$$
$$= -2.$$

(e) $\lim\limits_{x \to 0^+}\left(\frac{x}{a}\left[\frac{b}{x}\right] - \frac{b}{x}\left[\frac{x}{a}\right]\right)$, $a > 0$, $b > 0$.

Since $b > 0$, we can write $\frac{b}{x} = n + r$, where $n \le \frac{b}{x}$ is a non-negative integer and $0 \le r < 1$. Thus $\left[\frac{b}{x}\right] = n$, $x = \frac{b}{n + r}$, whence

$$\frac{x}{a}\left[\frac{b}{x}\right] = \frac{bn}{a(n + r)}$$
$$= \frac{b}{a(1 + \frac{r}{n})}.$$

As $x \to 0^+$, $n$ increases without bound and $\frac{r}{n} \to 0$. Since $a > 0$, $\left[\frac{x}{a}\right] = 0$ for $0 < x < a$. Thus,

$$\lim_{x \to 0^+}\left(\frac{x}{a}\left[\frac{b}{x}\right] - \frac{b}{x}\left[\frac{x}{a}\right]\right) = \lim_{x \to 0^+}\frac{x}{a}\left[\frac{b}{x}\right] - \lim_{x \to 0^+}\frac{b}{x}\left[\frac{x}{a}\right]$$
$$= \frac{b}{a} - \lim_{x \to 0^+} 0$$
$$= \frac{b}{a}.$$

(f) $\lim_{x \sim 0^-}\left(\frac{x}{a}\left[\frac{b}{x}\right] - \frac{b}{x}\left[\frac{x}{a}\right]\right)$, $a > 0$, $b > 0$.

The first term is similar to the first term in (e) except that $n$ is a negative integer. Since $a > 0$, $\left[\frac{x}{a}\right] = -1$ for $0 < |x| < a$; however, $\left|\frac{b}{x}\right|$ increases without bound as $x \sim 0^-$. Thus $\lim_{x \sim 0^-}\left(\frac{x}{a}\left[\frac{b}{x}\right] - \frac{b}{x}\left[\frac{x}{a}\right]\right)$ does not exist.

(g) $\lim_{x \sim 0^+} \frac{\sqrt{x}}{\sqrt{4 + \sqrt{x}} - 2}$.

$$\lim_{x \sim 0^+} \frac{\sqrt{x}}{\sqrt{4 + \sqrt{x}} - 2} = \lim_{x \sim 0^+} \frac{\sqrt{x}}{\sqrt{4 + \sqrt{x}} - 2} \cdot \frac{\sqrt{4 + \sqrt{x}} + 2}{\sqrt{4 + \sqrt{x}} + 2}$$

$$= \lim_{x \sim 0^+} \frac{\sqrt{x}}{\sqrt{x}} \cdot (\sqrt{4 + \sqrt{x}} + 2)$$

$$= 4.$$

## Teacher's Commentary

## Appendix 7

## CONTINUITY THEOREM

TC A7-1. Completeness of the Real Number System. The Separation Axiom.

The completeness of the real number system is a consequence of the Separation Axiom. We note that in the axiom the sets $A$ and $B$ might not be disjoint; e.g.,

$$A = \{x : x \leq 0\}, \; B = \{x : x \geq 0\}$$

and $0$ is the unique separation number. Observe that the word "separates" as used in the separation axiom does not mean that $s$ is not in either $A$ or $B$. The separation number $s$ may be in either set, both sets, or neither set.

The completeness of the field of real numbers plays a central role in the rigorous development of the calculus. We have based our logical development upon the Separation Axiom because of its intuitive geometric content. It asserts the absence of gaps (holes) in the real number line. The Least Upper Bound Principle and the Separation Axiom are logically equivalent. Thus, in the sequel, we can feel free to use whichever is most convenient.

We have approached the real numbers axiomatically; i.e., we have given a set of axioms rich enough to yield the properties that we need. An alternative approach is to define the set of real numbers in terms of a more basic set, say the rational numbers. This approach was taken in the nineteenth century by Dedekind, Cantor, and others. The original article of Dedekind now appears in the paperback, Essays in the Theory of Numbers, by R. Dedekind, Dover, 1963. His account is both lucid and elementary.

## Solutions Exercises A7-1

1. Prove Corollary 1 to the Least Upper Bound Principle. If $M$ is the least upper bound of the set $A$, then for each positive $\epsilon$ there exists an $\alpha \epsilon A$ such that $\alpha > M - \epsilon$.

   Suppose there is no $\alpha > M - \epsilon$. Then $M' : M - \epsilon < M' < M$ is an upper bound for $A$, contradicting $M$ as the least upper bound.

2. Prove Corollary 2 to the Least Upper Bound Principle. A set of numbers which is bounded below has a greatest lower bound.

   For the proof of Corollary 2, let $\overline{A}$ be a set which is bounded below and let $\overline{B}$ be the set of lower bounds of $\overline{A}$. The set $A = \{-\alpha : \alpha \epsilon \overline{A}\}$ has as the set of its upper bounds, $B = \{-\beta : \beta \epsilon \overline{B}\}$. The greatest lower bound $\overline{M}$ of $\overline{A}$ is given by $\overline{M} = -M$ where $M$ is the least upper bound of $A$.

3. (a) Consider the sets $A$ of positive rational numbers $\alpha$ satisfying $\alpha^2 < 2$, and $B$ of positive rational numbers $\beta$ satisfying $\beta^2 > 2$. Prove if $\alpha \epsilon A$ and $\beta \epsilon B$ that $\alpha < \beta$.

   If $\alpha = \beta$, then $\alpha^2 = \beta^2$ contradicting $\alpha^2 < \beta^2$. If $\alpha > \beta$, then since $\alpha > 0$ and $\beta > 0$, we have $\alpha^2 > \beta^2$. These contradictions force the conclusion $\alpha < \beta$.

   (b) Show that a separation number $s$ for the sets $A$ and $B$ must satisfy $s^2 = 2$; i.e., $s = \sqrt{2}$.

   Suppose $s$ is a separation number for $A$ and $B$ and $s \neq \sqrt{2}$, say $s = \sqrt{2} + \epsilon$, $\epsilon > 0$. Then, $\sqrt{2} + \frac{\epsilon}{2} < s$ but $\sqrt{2} + \frac{\epsilon}{2} \epsilon B$, since $(\sqrt{2} + \frac{\epsilon}{2})^2 = 2 + \epsilon\sqrt{2} + \frac{\epsilon^2}{4} > 2$. This contradiction indicates $s = \sqrt{2} + \epsilon$ is not a separation number. $s = \sqrt{2} - \epsilon$ can also be shown to fail, leaving only $s = \sqrt{2}$ as the separation number.

(c) Prove that $\sqrt{2}$ is irrational.

Assume $\frac{p}{q}$ is a rational solution in lowest terms of $x^2 = 2$.

Then, $p^2 = 2q^2$ implies $p^2$ is even and thus $p = 2r$, where $r$ is an integer. Now $4r^2 = 2q^2$ implies $q$ is even which contradicts our original assumption that $\frac{p}{q}$ is in lowest terms. Hence $\sqrt{2}$ is irrational.

4. (a) Prove for every real number $a$, that there is an integer $n$ greater than $a$ (Principle of Archimedes).

Recall that by definition each integer $n$ has a succeeding integer $n + 1$. Suppose there were no integer greater than $a$. Then, since the integers would have an upper bound $a$, they would have a least upper bound $M$. The number $M - \frac{1}{2}$ would not be an upper bound since $M$ is least. Consequently there is an integer $n > M - \frac{1}{2}$. Then $n + 1 > M + \frac{1}{2} > M$, contrary to the definition of least upper bound.

(b) Prove that given any $\epsilon > 0$ there is an integer $n$ such that $0 < \frac{1}{n} < \epsilon$.

Part (a) proved that there is an integer $n > \frac{1}{\epsilon}$. Hence $\epsilon > \frac{1}{n} > 0$.

5. (a) We define the infinite decimal

$$c_0 . c_1 c_2 c_3 \ldots ,$$

where $c_0$ is an integer, and $c_1, c_2, c_3, \ldots,$ are digits as the number $r$ where

$$c_0 + \frac{c_1}{10} + \frac{c_2}{10^2} + \ldots + \frac{c_n}{10^n} \leq r < c_0 + \frac{c_1}{10} + \frac{c_2}{10^2} + \ldots + \frac{c_n + 1}{10^n}.$$

Show that the preceding inequality does, in fact, define a unique real number.

Define two sets, $A_1$ the set of all $a_n$, and $B_1$ the set of all $b_n$, so that

$$a_n = c_0 + \frac{c_1}{10} + \dots + \frac{c_n}{10^n}$$

$$b_n = c_0 + \frac{c_1}{10} + \dots + \frac{c_n + 1}{10^n}.$$

We have $a_n \leq r < b_n$ for all $n$. Then $r$ is a separation number for $A$, and $B$. Given any $\epsilon > 0$, then $b_n - a_n = \frac{1}{10^n} < \epsilon$ for $n$ sufficiently large. By Lemma A7-1, $r$ is the unique separation number for $A$ and $B$.

(b) Given a real number $r$ we define its decimal representation recursively in terms of the integer part function $[x]$ as follows:

$$c_0 = [r]$$

$$c_n = \left[10^n\left(r - c_0 - \frac{c_1}{10} - \frac{c_2}{10^2} - \dots - \frac{c_{n-1}}{10^{n-1}}\right)\right].$$

Show that the inequality in part (a) is satisfied for this choice of $c_n$. Show also that decimals consisting entirely of 9's from some point on are avoided. (Thus, we obtain $2 = 2.000 \dots$ but not $2 = 1.999 \dots$)

Since $x - 1 < [x] \leq x$,

(1) $$10^n\left(r - c_0 - \dots - \frac{c_{n-1}}{10^{n-1}}\right) - 1 < c_n$$

$$\leq 10^n\left(r - c_0 - \dots - \frac{c_{n-1}}{10^{n-1}}\right),$$

or equivalently

(2) $$c_0 + \dots + \frac{c_n}{10^n} \leq r < c_0 + \dots + \frac{c_n + 1}{10^n}.$$

(Note the strong inequality on the right.)

Next, we establish for $n \geq 1$ that $c_n$ is a digit by mathematical indication.

For $c_1$, we have from (1) that $10(r - c_0) - 1 < c_1 \leq 10(r - c_0)$ where $0 \leq (r - c_0) < 1$.

Hence, $-1 < c_1 < 10$

or, since $c_1$ is an integer $\geq 0$, $0 \leq c_1 \leq 9$. Now suppose that $c_n$ is a digit $0 \leq c_n \leq 9$. In the notation of Number 5a, we see from (1) that

(3) $$c_n \leq 10^n(r - a_{n-1}) < 1 + c_n .$$

Now, replacing $n$ by $n + 1$ in (1) we have

(4) $$10^{n+1}(r - a_{n-1} - \frac{c_n}{10^n}) - 1 < c_{n+1} \leq 10^{n+1}(r - a_{n-1} - \frac{c_n}{10^n}).$$

Using the inequality (3) in (4) we obtain

$$-1 < c_{n+1} < 10,$$

from which we conclude that $c_{n+1}$ is a digit.

Now, let us suppose that $r$ can be represented as a decimal with an infinite string of 9's

$$r = d_0 . d_1 d_2 \cdots d_{p-1} d_p 999 \ldots$$

where we may without loss of generality suppose that either $p = 0$ or $d_p \neq 9$ (i.e., that the last decimal place where a 9 does not appear, if there is any, is the p-th place). Consider the number

$$\rho = d_0 . d_1 d_2 \cdots (1 + d_p) 000 \ldots .$$

Take $d_q = 9$ for $q > p$. For any index $q$, then, we see that both $\rho$ and $r$ lie between the numbers

$$d_0 + \frac{d_1}{10} + \ldots + \frac{d_q}{10^q}$$

and $$d_0 + \frac{d_1}{10} + \ldots + \frac{1 + d_q}{10^q} .$$

Since the upper and lower estimates here differ by $\frac{1}{10^q}$, it follows that $\rho = r$. Now the method of representation given above yields a unique decimal, since (2) is equivalent to the definition of $d_n$ by means of the integer part function. Taking $c_q = d_q$ for $q < p$, $c_p = 1 + d_p$, and $c_q = 0$ for $q > p$ we see that equality holds on the left in (2) for $q \geq p$; hence any other representation then the terminating one is precluded.

6. An infinite decimal $c_0 . c_1c_2c_3 \cdots$, is said to be periodic if for some fixed value $p$, the period of the decimal, we have $c_{n+p} = c_n$ for all $n$ satisfying $n \geq n_0$, where we require that $p$ is the smallest positive integer satisfying this condition. In words, from some place on, the decimal consists of the indefinite repetition of the same $p$ digits. Thus

$$\frac{1}{3} = .33333\cdots$$

$$\frac{15}{44} = .34090909\cdots$$

are periodic decimals. It is convenient to indicate a cycle of $p$ digits by underlining, rather than repetition; e.g.,

$$\frac{22}{7} = 3.\underline{142857}\ .$$

(a) Prove that every periodic decimal represents a rational number. (Hint: Consider the decimal as a geometric progression.)

Let $\quad r = c_0 . c_1 \cdots c_q \underline{b_1b_2 \cdots b_p}$

Then $\quad r = \dfrac{\gamma}{10^q} + \dfrac{\beta}{10^{p+q}} + \dfrac{\beta}{10^{2p+q}} + \dfrac{\beta}{10^{3p+q}} + \cdots$

where $\quad \gamma = 10^q . (c_0 . c_1c_2 \cdots c_q)$.

and $\quad \beta = 10^p . (0 . b_1b_2 \cdots b_p)$.

This latter representation for $r$ is an infinite geometric series with common ratio $\dfrac{1}{10^p}$. Whence we have

$$r = \frac{\gamma}{10^q} + \frac{\beta}{10^{p+q}(1 - 10^{-p})}$$

$$= \frac{(10^p - 1)\gamma + \beta}{10^q(10^p - 1)}$$

which is a rational number, since $(10^p - 1)\gamma + \beta$ and $10^q(10^p - 1)$ are integers.

(b) Prove that every rational number has a periodic decimal representation. (A "terminating" decimal in which each place beyond a certain point is zero is considered as a special case of periodic decimals.) If $r = \frac{s}{t}$ represents a rational number given in lowest terms, find the largest possible period of the infinite decimal representation of $r$ in terms of the denominator $t$.

From (a) and (b) we conclude that a decimal which is not periodic represents an irrational number, and conversely.

We first submit a specific case which contains the germ of the proof: we discuss the rational number $\frac{8}{7}$.

$$\frac{8}{7} = 1 + \frac{1}{7}$$

$$\frac{1}{7} = \frac{1}{10}\left(\frac{10}{7}\right) = \frac{1}{10}\left(1 + \frac{3}{7}\right),$$

$$\frac{3}{7} = \frac{1}{10}\left(\frac{30}{7}\right) = \frac{1}{10}\left(4 + \frac{2}{7}\right),$$

$$\frac{2}{7} = \frac{1}{10}\left(\frac{20}{7}\right) = \frac{1}{10}\left(2 + \frac{6}{7}\right),$$

$$\frac{6}{7} = \frac{1}{10}\left(\frac{60}{7}\right) = \frac{1}{10}\left(8 + \frac{4}{7}\right),$$

$$\frac{4}{7} = \frac{1}{10}\left(\frac{40}{7}\right) = \frac{1}{10}\left(5 + \frac{5}{7}\right),$$

$$\frac{5}{7} = \frac{1}{10}\left(\frac{50}{7}\right) = \frac{1}{10}\left(7 + \frac{1}{7}\right).$$

Compare this procedure with the "long division" process:

$$\begin{array}{r} .142857 \\ 7\overline{)1.000000} \\ \underline{7}\phantom{00000} \\ 30\phantom{0000} \\ \underline{28}\phantom{0000} \\ 20\phantom{000} \\ \underline{14}\phantom{000} \\ 60\phantom{00} \\ \underline{56}\phantom{00} \\ 40\phantom{0} \\ \underline{35}\phantom{0} \\ 50 \\ \underline{49} \\ 1 \end{array}$$

Note that we stop with a remainder 1 since it occured before. We would get a repetition of the same digits if we were to continue the division process:

$$\frac{8}{7} = 1.142857142857\ldots .$$

In general we shall always get a remainder which is repeated since there are only a finite numbers of different possible remainders. The argument given here is based on the division algorithm which must eventually repeat itself because remainders must recur.

Let $r = \frac{\sigma}{\tau}$ where $\sigma$ and $\tau$ are relatively prime and $\tau > 0$. Let the decimal expansion of $r$ as given by the method of Number 5 be

$$c_0 \,.\, c_1 c_2 c_3 \cdots .$$

Represent $\tau$ in the form $2^a 5^b m$ where $m$ has no factors of 2 or 5. Set $q = \max\{a, b\}$, $t = 10^q m$ and rewrite $r$ in the form

$$r = \frac{s}{t} = \frac{s}{10^q m}.$$

Observe that $m$ and $s$ are relatively prime.

If $m = 1$ we have exactly

$$r = \frac{s}{10^q} = c_0 \,.\, c_1 c_2 \cdots c_q$$

so that $c_n = 0$ for $n > q$ and the period is 1. If $m \neq 1$, take

$$\lambda_k = 10^{q+k} (c_0 \,.\, c_1 c_2 \cdots c_{q+k}).$$

From (2) in the solution of Number 5, we have

$$\lambda_k \leq \frac{10^k s}{m} < \lambda_k + 1,$$

whence

$$0 \leq 10^k s - m\lambda_k < m.$$

Consequently, on dividing $10^k s$ by $m$ we obtain the quotient $\lambda_k$ and the remainder $r_k = 10^k s - m\lambda_k$. Now $r_k \neq 0$ since $m$ and $10^k s$ are relatively prime. Thus $r_k$, as a remainder on division by $m$ can only be one of the integers 1, 2, ..., $m - 1$. It follows that at least two of the $m$ numbers $r_k$, for $k = 1, 2, \ldots, m$, must be the same; say $r_i = r_j$ with $j > i$. From this we now prove that the decimal expansion for $r$ is periodic with the period $p = j - i$, namely,

$$r = c_0 \,.\, c_1 c_2 \cdots c_{q+i} \underline{c_{q+i+1} \cdots c_{q+i+p}}.$$

We show first that $r_k = r_{k+p}$. Observe that,

$$r_{k+1} = 10^{k+1} s - m\lambda_{k+1} = 10^{k+1} s - m(10\lambda_k + c_{q+k+1})$$

$$= 10r_k - m\, c_{q+k+1}.$$

But $$c_{q+k+1} = \left[10^{q+k+1}(r - c_0 . c_1 c_2 \ldots c_{q+k})\right]$$

$$= \left[10\left(\frac{10^k s}{m} - \lambda_k\right)\right]$$

$$= \left[\frac{10r_k}{m}\right].$$

Combining the last two results, we obtain a representation for $r_{k+1}$ in terms of $r_k$ alone. It follows from $r_i = r_j$ that $r_{i+1} = r_{j+1}$, $r_{i+2} = r_{j+2}$, etc. Thus $r_k$ is periodic with period p. Since $c_{q+k+1}$ is a function of $r_k$ we see that the decimal also is periodic with period p.

(c) Prove for every positive prime $\alpha$ other than 2 and 5 that there exists an integer, all of whose digits are ones, for which $\alpha$ is a factor; i.e., $\alpha$ is a factor of some number of the form

$$10^n + 10^{n-1} + 10^{n-2} + \ldots + 10 + 1.$$

Let $\alpha$ be the given prime. We can write (from part (a))

$$\frac{1}{\alpha} = \frac{(10^p - 1)\gamma + \beta}{10^q(10^p - 1)}$$

or

$$\alpha\big((10^p - 1)\gamma + \beta\big) = 10^q(10^p - 1).$$

Since $\alpha$ is neither 2 nor 5 it follows that $\alpha$ is a factor of

$$10^p - 1 = 9(10^{p-1} + 10^{p-2} + \ldots + 10 + 1).$$

If $\alpha \neq 3$, then $\alpha$ must be a factor of the expression in parenthesis. If $\alpha = 3$, then $\alpha$ is a factor of $10^2 + 10 + 1$. In either case, the result is proved.

7. (a) Consider a polynomial with integer coefficients:

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 . \qquad (a_n \neq 0).$$

Prove that if $\frac{p}{q}$ is a rational root of this polynomial given in lowest terms, then $p$ is a factor of $a_0$ and $q$ is a factor of $a_n$.

If $\frac{p}{q}$ is a root of the polynomial, then

$$a_n \frac{p^n}{q^n} + a_{n-1} \frac{p^{n-1}}{q^{n-1}} + \dots + a_1 \frac{p}{q} + a_0 = 0,$$

whence, on multiplying by $q^n$,

$$a_n p^n + a_{n-1} p^{n-1} q + \dots + a_1 p q^{n-1} + a_0 q^n = 0.$$

It follows that $p$ is a factor of $a_0 q^n$ and $q$ is a factor of $a_n p^n$. Since $p$ and $q$ have no factors in common, we conclude that $p$ is a factor of $a_0$ and $q$ a factor of $a_n$.

(b) Show that $x^3 + x + 1$ has no rational root.

By the preceding result the only conceivable rational roots are $1$ and $-1$ and neither is a root.

(c) Prove that if $\sqrt{n}$ is rational then it is integral.

A rational root $\frac{p}{q}$ of $x^2 - n = 0$ must be an integer divisor of $n$, since $q = \pm 1$.

(d) Prove that $\sqrt{3} - \sqrt{2}$ is irrational.

Set $a = \sqrt{3} - \sqrt{2}$. Squaring we obtain

$$a^2 = 5 - 2\sqrt{6}$$

whence

$$(a^2 - 5) = -2\sqrt{6}$$

and

$$(a^2 - 5)^2 = 24$$

or

$$a^4 - 10a^2 + 1 = 0.$$

The only conceivable rational roots of this equation are $\pm 1$ and neither is a root.

<u>Alternatively</u>. Assume $\sqrt{3} - \sqrt{2} = r$ (rational); then,

$$r^2 - 2r\sqrt{2} + 2 = 3.$$

Since this implies that $\sqrt{2}$ is rational, $\sqrt{3} - \sqrt{2}$ is irrational.

## Solutions Exercises A7-2

1. Let $f(x)$ $$f(x) = \begin{cases} \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$$

Show that $f$ satisfies the conclusion of Theroem 8-2a on any interval $[0,b]$ but $f$ is not continuous at $x = 0$.

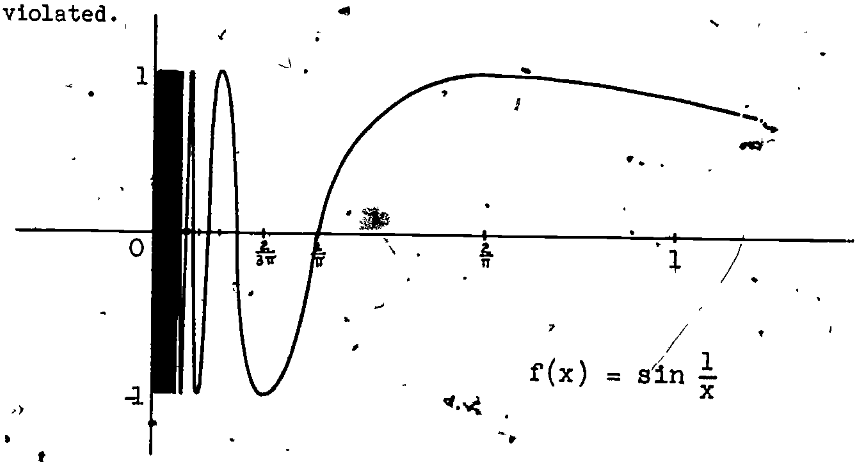
$g(y) = \sin y$ is a periodic function of period $2\pi$; so it will take on all of the values $-1 \leq g(y) \leq 1$ on any interval $n\pi \leq y \leq (n+2)\pi$. Certainly it will take on all of these values on the set of all $y$ such that $y \geq \frac{1}{b}$. In fact, each value of $g(y)$ is repeated as many times as we like on this set.

What we have said implies that $f(x) = \sin \frac{1}{x}$ will take on all the values $-1 \leq f(x) \leq 1$ on the interval $[0,b]$, and, in fact, $f(x)$ oscillates between $-1$ and $+1$ an unlimited number of times in any interval $[0,b]$, no matter how small.

This is a statement both that $f$ satisfies the conclusion of Theorem 8-2a, and that $f$ is not continuous at $x = 0$.

For: Theorem 8-2a states that $f$, under certain conditions, will assume all values between $0$ and $f(b)$ on $[0,b]$. We have seen that $f$ in fact assumes all values between $-1$ and $+1$.

Continuity of $f$ at $0$ implies that for any $\epsilon > 0$, we can find a $\delta > 0$ such that for any $x$: $0 < |x| < \delta$, $|f(x) - f(0)| < \epsilon$. But we have seen that no matter how small we make $\delta$, we can find values of $x$, $0 < |x| < \delta$, so that $f(x)$ takes on all values $-1 \leq f(x) \leq 1$, and $|f(x) - f(0)|$ takes on all values between $0$ and $1$ inclusive. We need only choose $\epsilon < 1$, and the condition for continuity is violated.




$f(x) = \sin \frac{1}{x}$

2. Prove that if $f$ is continued and has an inverse on $[a,b]$ and $f(a) < f(b)$ then $f$ is strictly increasing.

"$f$ has an inverse $f'$" means that $f^{-1}(y_0)$ is uniquely determined. Formally, if $y_0 \neq y_1$, then $f(y_0) \neq f(y_1)$.

Suppose $f$ is not strictly increasing that is, for some pair of $x_0, x_1$ in $[a,b]$

$$x_0 < x_1, \; f(x_1) \leq f(x_0).$$

We will show that there is an $x_2$ in $[a,b]$ such that $x_2 \neq x_0$, $f(x_2) = f(x_0)$, which contradicts the assumption that $f$ has an inverse. Consider the interval $[x_1,b]$.

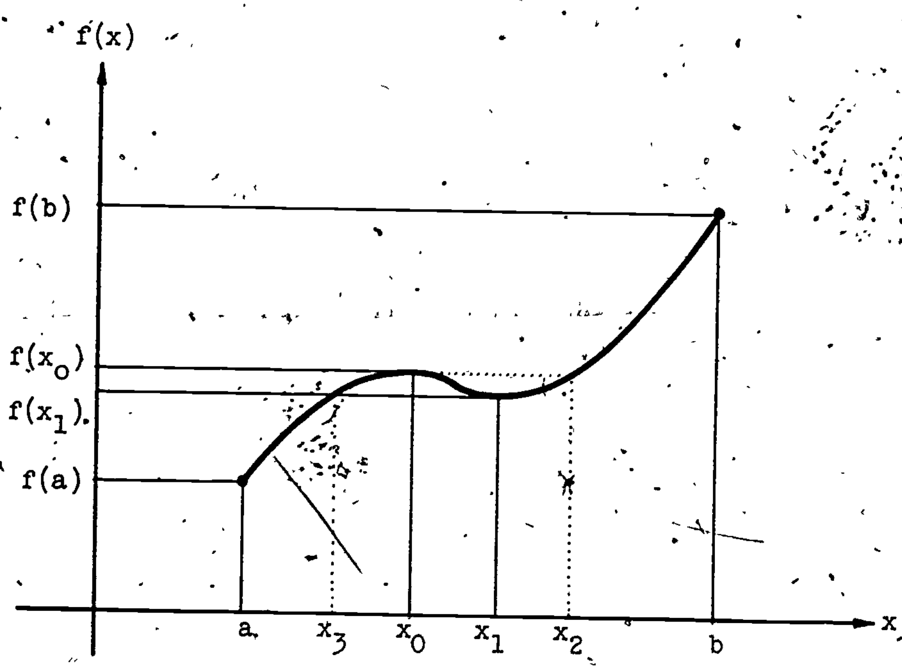
$$f(x_1) \leq f(x_0) = c \leq f(b).$$

By Theorem 8-2a, there is a value $x_2$ in $[x_1,b]$ such that

$$f(x_2) = c = f(x_0).$$

Certainly $x_2$ is in $[a,b]$, $x_2 > x_0$.

Similarly, we could show there is an $x_3$ in $x_3$ in $[a,x_0]$ such that $f(x_3) = f(x_1)$.

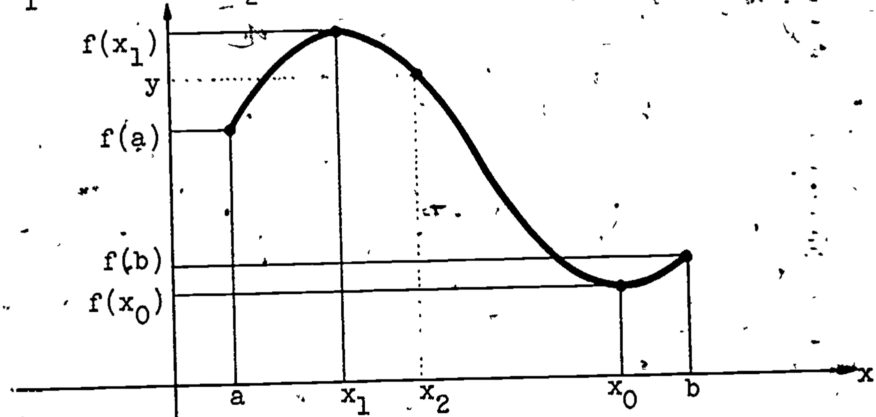


3. Prove that, if $f$ is continuous on $[a,b]$ then the image of $[a,b]$ is a closed interval.

From Theorem 8-2b, there are two points $x_0$ and $x_1$ with $a \le x_0 \le b$ and $a \le x_1 \le b$ such that

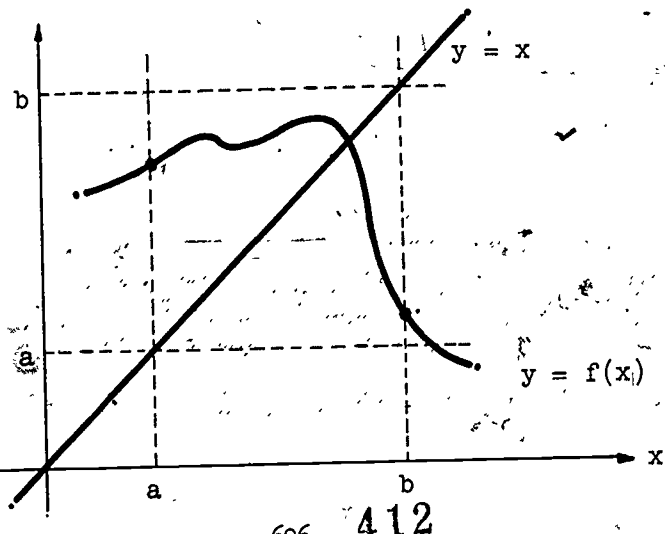
$$f(x_0) \le f(x) \le f(x_1) \quad \text{for all } x, \ a \le x \le b.$$

The image of $f$, then, is clearly a subset of the closed interval $[f(x_0), f(x_1)]$ and includes the endpoints. Theorem 8-2a tells us that the image of $f$ includes all the points in $[f(x_0), f(x_1)]$: (If we choose any value $y$, $f(x_0) < y < f(x_1)$, then there is a point $x_2$ between $x_0$ and $x_1$ such that $f(x_2) = y$.)

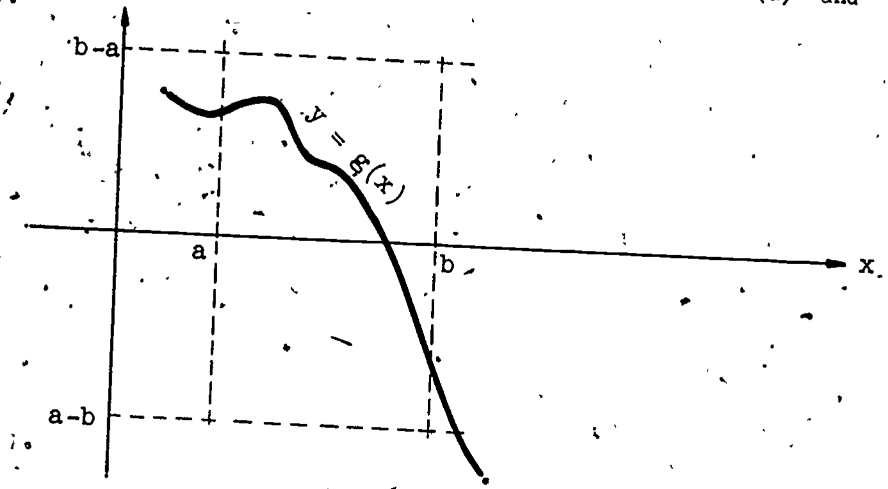



4. Prove that if $f$ is continuous in $[a,b]$ and all values of $f$ are in $[a,b]$ then there is an $x$ on $[a,b]$ for which $f(x) = x$.

We could interpret this problem as proving that the graph of the function $f : x \to f(x)$ intersects the graph of the function $g : x \to x$



Note that this is analogous to the situation in Theorem 8-2a. In Theorem 8-2a, the line intersected was $y = d$, $d$ between $f(a)$ and $f(b)$.




Consider the function

$$g(x) = f(x) - x.$$

This is clearly a continuous function, since it is the difference between two continuous functions. We wish to show that $g(x) = 0$ for some $x$ in $[a,b]$.

(1) Suppose $g(x) > 0$ for all $x$ in $[a,b]$.

Then $g(b) = f(b) - b > 0$, which is to say $f(b) > b$, which contradicts the assumption that the image of $f$ is enclosed in $[a,b]$.

(2) Suppose $g(x) < 0$ for all $x$ in $[a,b]$.

Similarly, this leads to the contradiction $f(a) < a$.

(3) Suppose $g(x)$ takes on both positive and negative values in $[a,b]$.

Choose $c$, $d$ so that $g(c) < 0 < g(d)$ or $g(d) < 0 < g(c)$. Since $g$ is continuous, Theorem 8-2a applies on $[c,d]$. Therefore we can find a point $x_0$ in $[c,d]$, such that $g(x_0) = 0$.

5. Suppose

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Does $f$ satisfy the hypotheses of Theorem 8-2b on the interval $[0,1]$? Does (8) hold for $f$ on $[0,1]$? on $[10^{-100},1]$?

$f$ does not satisfy the hypotheses of Theorem 8-2b, since it is not continuous at $x = 0$. (See Problem 1.)

(8) does not hold on $[0,1]$. $f(x) = \frac{1}{x}$ can be made as large as we like if we choose $x$ sufficiently close to $x = 0$.

(8) holds on $[10^{-100},1]$. $\text{lub}\ f(x) = 10^{100}$ on this interval.

6. Is the continuity of $f$ essential to the hypothesis of (8)?

Yes. Consider the example of Exercise 5.

7. Can a discontinuous function whose domain is a closed interval be bounded?

Yes. Consider $x \to [2x]$ on $[0,1]$, or $x \to [nx]$.

8. Do Numbers 6 and 7 amount to the same question?

No. Number 6 asks, Is Theorem 8-2b true if we drop the hypothesis of continuity? while Number 7 asks, Is continuity necessary for boundedness?

9. Can a nonconstant function whose domain is the set of real numbers be bounded?

Yes; e.g., $x \to \sin x$ or $x \to \dfrac{x^2}{x^2+1}$.

10. Show that a function $f$ which is increasing in some neighborhood of each point of an interval $[a,b]$ is an increasing function in $[a,b]$.

Consider the set $A$ of points $t$ in $[a,b]$ such that $f$ is increasing in $[a,t]$. Call $\alpha$ the least upper bound of $A$. Then for $\beta > \alpha$, $f$ is not increasing in $(a,\beta)$.

We are given that $f$ is increasing in a neighborhood of $\alpha$ if $\alpha < b$. Therefore, for some $h$, $f$ is increasing in $(\alpha - h, \alpha + h)$. Hence, $f$ is increasing in $(a, \alpha + h)$. But this means that $\alpha + h$ is in $A$, while $\alpha$ is an upper bound of $A$. So $\alpha \geq b$ and $f$ is increasing in $(a,b)$.

11. A function has the property that for each point of an interval where it is defined, there is a neighborhood in which the function is bounded. Show that the function is bounded over the whole interval. (This is an example where a local property implies a global one. It is clear that the global property here implies the local one.)

Let $I$ be the interval for which $f$ is locally bounded and let $a$ and $b$ be the respective left and right endpoints of $I$ (no implication that $I$ is either open or closed). Let $A$ be the set of points consisting of the point $a$ and those points $\alpha$ of $I$ for which $f(x)$ is bounded on the interval $I_\alpha = I \quad \{x : x \leq \alpha\}$. Take $\bar{\alpha} = \sup A$. If $\alpha > \bar{\alpha}$, then $f$ cannot be bounded on $I_\alpha$. If $\bar{\alpha} = b$, then $f$ is bounded on $I$. If $\bar{\alpha} < b$, then $f$ is bounded on a neighborhood of $\bar{\alpha}$. It follows that $f$ is bounded on the union of $A$ and this neighborhood, contradicting that there is no interval $I_\alpha$ with $\alpha > \bar{\alpha}$ for which $f$ is bounded.

# Solutions Exercises A7-3

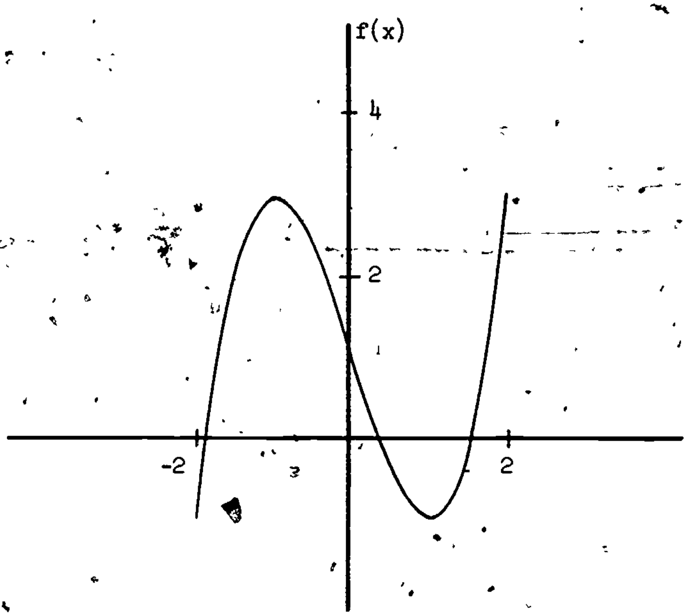
1. Prove Corollary 2 to Lemma A7-3.

   Corollary 2. A polynomial of degree $n$ can have no more than $n$ distinct real roots.

   Let $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, $a_n \neq 0$. The n-th derivative of $p$ is given by $p^{(n)}(x) = n!a_n$, a nonzero constant function. It follows by Corollary 1 to Lemma A7-3 that $p^{(n-1)}$ has at most one real root. Applying this argument recursively we obtain the desired result.

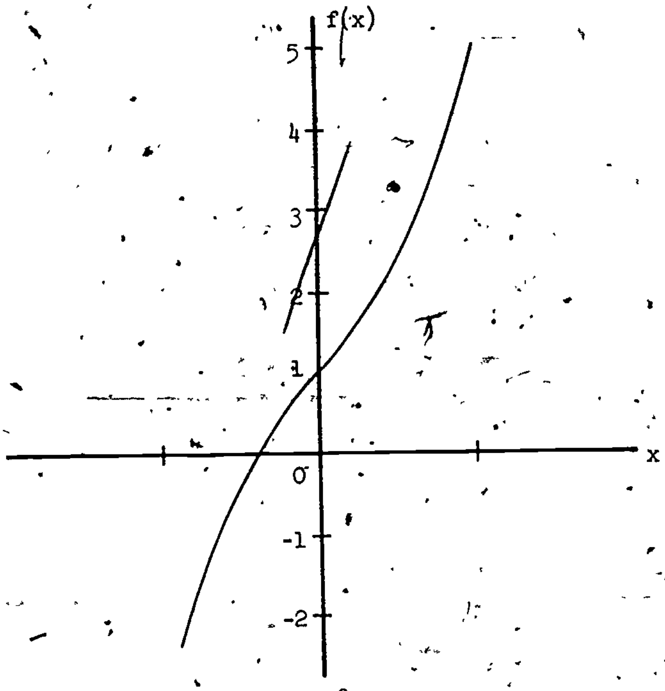
2. Sketch the graphs of the functions in Example A7-3a.

(i)



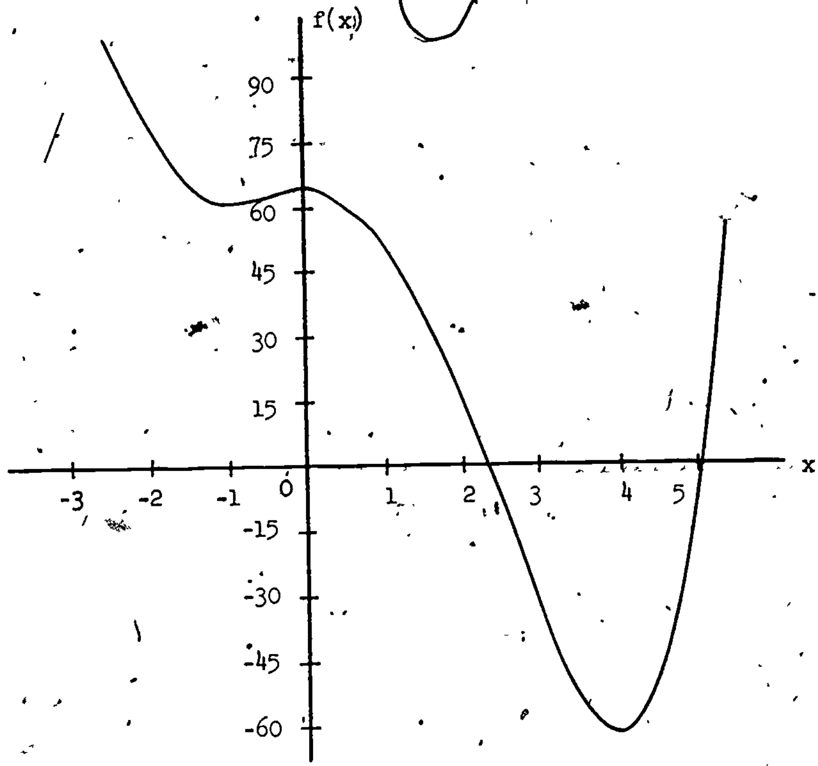

$f : x \to x^3 - 3x + 1$

(ii)




$f : x \longrightarrow x^3 + 3x + 1$

(iii)




$f : x \longrightarrow x^4 - 4x^3 - 8x^2 + 64$

3. Is the following converse of Rolle's Theorem true? If $f$ is continuous on the closed interval $[p,q]$ and differentiable on the open interval $(p,q)$, and if there is at least one point $u$ in the open interval where $f'(u) = 0$, then there are two points $m$ and $n$ where $p \leq m < u < n \leq q$ such that $f(m) = f(n)$.

Not true. Counterexample: $y = x^3$ for any interval containing $x = 0$ in its interior.

4. Does Rolle's Theorem justify the conclusion that $\frac{dy}{dx} = 0$ for some values of $x$ in the interval $-1 \leq x \leq 1$ for $(y + 1)^3 = x^2$?

If $(y + 1)^3 = x^2$, $\frac{dy}{dx} = \frac{2x}{3(y + 1)^2} = \frac{2}{3x^{1/3}}$. The conclusion of Rolle's Theorem does not hold for the closed interval $[-1,1]$ since $\frac{dy}{dx}$ does not exist at $x = 0$.

5. Given: $f(x) = x(x - 1)(x - 2)(x - 3)(x - 4)$. Determine how many solutions $f'(x) = 0$ has and find intervals including each of these without calculating $f'(x)$.

By Corollary 1 to Lemma A7-3, $f'(x) = 0$ has four solutions. There is one solution in each of the open intervals $(0,1)$, $(1,2)$, $(2,3)$, and $(3,4)$ since the zeros of $f$ are 0, 1, 2, 3, 4.

6. Verify that Rolle's Theorem (Lemma A7-3) holds for the given function in the given interval or give a reason why it does not.

(a) $f : x \to x^3 + 4x^2 - 7x - 10$, $[-1,2]$

(b) $f : x \to \frac{2 - x^2}{x^4}$, $[-1,1]$

(a)
$$f(x) = x^3 + 4x^2 - 7x - 10$$
$$f'(x) = 3x^2 + 8x - 7$$
$$f(-1) = f(2) = 0$$

One of the zeros of $f'$, $\frac{-4 + \sqrt{37}}{3}$, is in the interval $(-1,2)$. Thus, Rolle's Theorem holds.

(b) $f : x \to \frac{2 - x^2}{x^4}$

The function is not continuous at $x = 0$ and hence the theorem does not apply.

7. Prove that the equation

$$f(x) = x^n + px + q = 0$$

cannot have more than two real solutions for an even integer $n$ nor more than three real solutions for an odd $n$. Use Rolle's Theorem.

$$f(x) = x^n + px + q$$

$$f'(x) = nx^{n-1} + p = 0$$

If $n$ is even, $f'$ is of odd degree and $f'(x) = 0$ has one and only one solution. It follows from Corollary 1 to Lemma A7-3 that $f$ cannot have more than two real solutions in this case. Similarly, if $n$ is odd, $f'$ is of even degree and $f'(x) = 0$ has no more than two solutions. In this case $f(x) = 0$ has no more than three real solutions.

8. A function $g$ has a continuous second derivative on the closed interval $[a,b]$. The equation $g(x) = 0$ has three different solutions in the open interval $(a,b)$. Show that the equation $g''(x) = 0$ has at least one solution in the open interval $(a,b)$.

By Corollary 1 to Lemma A7-3, $g'$ has at least two zeros in the open interval $(a,b)$ and hence the derivative of $g'$, which is $g''$, must have at least one zero in the interval $(a,b)$.

9. Show that the conclusion of the Mean Value Theorem does not follow for $f(x) = \tan x$ in the interval $1.5 < x < 1.6$ and explain why.

The theorem does not apply. The function $f$ is not continuous on the open interval $(1.5,1.6)$. (Note that $1.5 < \frac{\pi}{2} < 1.6$ and $f(x)$ is not defined at $\frac{\pi}{2}$.) $\tan(1.5) > 0$, $\tan(1.6) < 0$. Therefore, if the Mean Value Theorem held in the interval $[1.5,1.6]$, there would exist a $u$ in $(1.5,1.6)$ such that $f'(u) < 0$. But $D \tan x = \sec^2 x$, which is postive.

10. For each of the following functions show that the Mean Value Theorem fails to hold on the interval $[-a,a]$ if $a > 0$. Explain why the theorem fails.

(a) $f : x \to |x|$

$f(-a) = f(a) = a$. Yet $f'(x)$ is either $+1$ or $-1$ and never zero, so the Mean Value Theorem fails to hold. ($f'(0)$ does not exits.)

(b) $f : x \to \frac{1}{x}$

If the Mean Value Theorem holds for $f$ on the interval $[-a,a]$; then for some $u$ in $(-a,a)$, $f'(u) = \frac{4}{a^2}$. But $f'(x) = -\frac{1}{x^2}$ and so is never positive. ($f$ is not continuous at $x = 0$.)

11. Show that the equation $x^5 + x^3 - x - 2 = 0$ has exactly one solution in the open interval $(1,2)$.

$f(x) = x^5 + x^3 - x - 2$ and $f'(x) = 5x^4 + 3x^2 - 1$. Since $f(1) < 0$ and $f(2) > 0$ the function $f$ has a zero in the interval $(1,2)$, by the Intermediate Value Theorem and since $f'(x) > 0$ for $x > 1$, the function $f$ has only one zero in the interval.

12. Show that $x^2 = x \sin x + \cos x$ for exactly two real values of $x$.

Let $f(x) = x^2 - x \sin x - \cos x$. Then

$$\begin{aligned} f'(x) &= 2x - x \cos x \\ &= x(2 - \cos x). \end{aligned}$$

$f'(x) = 0$ if and only if $x = 0$ and hence $f$ has no more than two zeros. Since $f(0) = -1$ and $f(\pi) = f(-\pi) = \pi^2 + 1$ we conclude that there are zeros in the open intervals $(-\pi,0)$ and $(0,\pi)$, by the Intermediate Value Theorem.

13. Find a number that can be chosen as the number $c$ in the Mean Value Theorem for the given function and interval.

(a) $f : x \to \cos x,\ 0 \le x \le \frac{\pi}{2}$

$c = \arcsin\left(\frac{2}{\pi}\right)$

(b) $f : x \to x^3, \quad -1 \le x \le 1$

$$c = -\frac{\sqrt{3}}{3} \text{ or } \frac{\sqrt{3}}{3}$$

(c) $f : x \to x^3 - 2x^2 + 1, \; -1 \le x \le 0$

$$c = \frac{2 - \sqrt{13}}{3}$$

(d) $f : x \to \cos x + \sin x; \; 0 \le x \le 2\pi$

$$c = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

14. Derive each of the following inequalities by applying the Mean Value Theorem.

(a) $|\sin x - \sin y| \le |x - y|$

If $f(x) = \sin x$, then $f$ is continuous and differentiable for all $x$. By the Mean Value Theorem, for any $x \neq y$,

$$\frac{\sin x - \sin y}{x - y} = \cos u, \quad \text{where } x < u < y,$$

and hence

$$\left|\frac{\sin x - \sin y}{x - y}\right| = |\cos u| \le 1,$$

or

$$|\sin x - \sin y| \le |x - y|.$$

(b) $\dfrac{x}{1 + x^2} < \arctan x < x \quad \text{if } x > 0.$

If $g(x) = \arctan x$, $g$ is continuous for $x \ge 0$ and differentiable for all $x > 0$. By the Mean Value Theorem,

$$\frac{\arctan x - \arctan 0}{x} = \frac{1}{1 + u^2}, \quad \text{where } 0 < u < x.$$

Since

$$\frac{1}{1 + x^2} < \frac{1}{1 + u^2} < 1,$$

we have

$$\frac{1}{1 + x^2} < \frac{\arctan x}{x} < 1,$$

so that

$$\frac{x}{1 + x^2} < \arctan x < x, \quad x > 0.$$

15. Use the Mean Value Theorem to approximate $\sqrt[3]{1.008}$.

Here $f(x) = \sqrt[3]{x}$ and we can choose $p = 1$, $q = 8$ for numerical simplicity. If we approximate $f(x)$ by the linear function $g$ whose graph is the chord joining the points $(1,1)$ and $(8,2)$,

$$g(x) = 1 + \frac{1}{7}(x - 1).$$

Thus, an approximate value of $\sqrt[3]{1.008}$ is given by $g(1.008) = 1.001$. Since $|g(x) - f(x)| \leq 2M_1(x - p)$ where $M_1$ is an upper bound of $f'(x)$ in $[p,q]$, we have the error estimate

$$|1.001 - \sqrt[3]{1.008}| \leq 2(.008) \cdot \max \frac{1}{3x^{2/3}}$$

or

$$|1.001 - \sqrt[3]{1.008}| < 2(.008)(\tfrac{1}{3}) < .0054$$

and

$$0.9957 < \sqrt[3]{1.008} < 1.0064.$$

To get a better approximation, we choose a number $q$ closer to 1. A convenient value is $q = \frac{27}{8}$. Then

$$g(x) = 1 + \frac{\frac{3}{2} - 1}{\frac{27}{8} - 1}(x - 1) = 1 + \frac{4}{19}(x - 1),$$

and

$$g(1.008) = 1 + \frac{4}{19}(.008) \approx 1.0017.$$

Here

$$|1.0017 - \sqrt[3]{1.008}| \leq 2(.008)\frac{1}{3}$$

and

$$.9963 < \sqrt[3]{1.008} < 1.0071.$$

Combining this with the previous approximation, we have

$$.9963 < \sqrt[3]{1.008} < 1.0064.$$

We can sharpen the upper bound by choosing $q$ very close to 1. Get arbitrarily close to $\sqrt[3]{1.008}$.

Alternatively, we can proceed as follows. From the Mean Value Theorem

$$\frac{f(q) - f(p)}{q - p} = f'(c), \qquad p < c < q,$$

it follows that

$$\frac{\sqrt[3]{1.008} - \sqrt[3]{1}}{1.008 - 1} = \frac{1}{3\sqrt[3]{c^2}}, \qquad 1 < c < 1.008,$$

(here $f(x) = \sqrt[3]{x}$, $q = 1.008$, $p = 1$). Then, since $f'(c)$ is monotonic in $[1,1.008]$, we have

$$\frac{.008}{3\sqrt[3]{\frac{27}{8}}} < \frac{.008}{3\sqrt[3]{1.008^2}} < \sqrt[3]{1.008} - 1 < \frac{.008}{3}.$$

or

$$1.0017 < \sqrt[3]{1.008} < 1.0027.$$

16. Use the Mean Value Theorem to approximate $\cos 61^\circ$.

$$\frac{\cos q - \cos p}{q - p} = -\sin c, \qquad p < c < q.$$

Choose

$$p = \frac{\pi}{3}, \quad q = \frac{\pi}{3} + \frac{\pi}{180}.$$

Note that $p$ and $q$ are in radians and $1^\circ = \frac{\pi}{180}$ radians.

$$\frac{\cos(\frac{\pi}{3} + \frac{\pi}{180}) - \cos\frac{\pi}{3}}{\frac{\pi}{180}} = -\sin c.$$

Since

$$\sin 60^\circ < \sin c < 1,$$

$$-\frac{\pi}{180}\cdot 1 < \cos 61^\circ - \frac{1}{2} < -\frac{\pi}{180}\cdot\frac{1}{2}$$

or

$$\frac{1}{2} - \frac{\pi}{180} < \cos 61^\circ < \frac{1}{2} - \frac{\pi}{360}.$$

17. Show that $a\left(1 + \frac{\epsilon}{n(a^n + \epsilon)}\right) < \sqrt[n]{a^n + \epsilon} < a\left(1 + \frac{\epsilon}{na^n}\right)$ for $\epsilon > 0$, $a > 1$, $n > 1$ ($n$ rational).

Let $f(x) = \sqrt[n]{x}$. Then

$$\frac{f(q) - f(p)}{q - p} = f'(u), \qquad p < u < q.$$

Choose $q = a^n + \epsilon$, $p = a^n$. Thus,

$$\frac{\sqrt[n]{a^n + \epsilon} - \sqrt[n]{a^n}}{\epsilon} = \frac{u^{1/n}}{n \cdot u}, \quad a^n < u < a^n + \epsilon$$

and if $a > 1$,

$$\sqrt[n]{a^n + \epsilon} < \sqrt[n]{a^n + \epsilon} - a < \frac{\epsilon}{na^n}.$$

Since

$$\frac{\epsilon \sqrt[n]{a^n + \epsilon}}{n(a^n + \epsilon)} > \frac{\epsilon a}{n(a^n + \epsilon)},$$

we obtain the desired inequalities.

18. Using Number 17, obtain the following approximations:

(a) $3 + \frac{1}{10} < \sqrt[3]{30} < 3 + \frac{1}{9}$.

If $n = a = \epsilon = 3$ in the result of Number 17, then

$$3 + \frac{1}{10} < \sqrt[3]{30} < 3 + \frac{1}{9}.$$

(b) $3 + \frac{3}{5(244)} < \sqrt[5]{244} < 3 + \frac{1}{405}$.

If we set $n = 5$, $a = 3$, $\epsilon = 1$ in the result of Number 17, we obtain the desired result.

(c) Show that the approximation

$$\frac{1}{2}\left(3 + \frac{3}{5(244)} + 3 + \frac{1}{405}\right) \text{ to } \sqrt[5]{244}$$

is correct to at least five decimal places.

We note that, in general, if $a < b < c$, then the error in taking $\frac{a + c}{2}$ as an approximation to $b$ is no greater than $\frac{1}{2}(c - a)$. Letting $a = 3 + \frac{3}{5(244)}$, $b = \sqrt[5]{244}$, and $c = 3 + \frac{1}{405}$,

$$\frac{1}{2}\left(\frac{1}{405} - \frac{3}{5(244)}\right) = \frac{(1220 - 1215)}{2(405)(1220)} = \frac{1}{2(81)(1220)} < \frac{1}{100,000}.$$

Therefore,

$$\left|\frac{1}{2}\left(3 + \frac{1}{405} + 3 + \frac{3}{5(244)}\right) - \sqrt[5]{244}\right| < 10^{-5}.$$

19. (a) Show that a straight line can intersect the graph of a polynomial function of n-th degree at most $n$ times.

Let $y = mx + b$ be an equation of a straight line and $p(x) = a_0 + a_1x + \ldots + a_nx^n$ be an n-th degree polynomial $(a_n \neq 0)$. Then if $y = p(x)$, we have $g(x) = 0$, where $g(x) = (a_0 - b) + (a_1 - m)x + a_2x^2 + \ldots + a_nx^n$, $a_n \neq 0$. By Corollary 2 to Lemma A7-3 (proved in Solution No. 1), $g(x)$, a polynomial of n-th degree, has at most $n$ distinct real roots. Therefore, a straight line can intersect the graph of a polynomial function of n-th degree at most $n$ times (unless, of course, $g(x)$ is identically 0; i.e., $p$ is linear and $p(x) = mx + b$).

(b) Obtain the corresponding result for rational functions.

Let $R(x) = \frac{f(x)}{g(x)}$ where $f$ is of degree $t \geq 0$ and $g$ of degree $s \geq 0$, $g(x) \neq 0$. When $R(x) = p(x)$, where $p$ is a polynomial function of degree $n$, then $p(x) \cdot g(x) - f(x) = 0$. Let $q(x) = p(x) \cdot g(x) - f(x)$. Then the degree of $q$ is at most $\max\{t, s + n\}$. Thus the graph of a rational function $x \to R(x) = \frac{f(x)}{g(x)}$, where $f$ is of degree $t \geq 0$ and $g$ is of degree $s \geq 0$, can intersect the graph of a polynomial of degree $n$ in at most $\max\{t, s + n\}$ points. Since $n = 1$ for the linear function $p$, we conclude that the graph of a straight line can meet the graph of $R$ in at most $\max\{t, s + 1\}$ points. (Again, if $R$ is linear there is an exceptional case.)

(c) Could $\sin x$ or $\cos x$ be rational functions? Justify your answer.

No. By the results of (b), if $\sin x$ were a rational function, then the equation $\sin x = 0$ would have a finite number of solutions. The same remark applies to $\cos x$.

20. Prove the intermediate value property for derivatives; namely, if $f$ is differentiable on the closed interval $[p,q]$ then $f'(x)$ takes on every value between $f'(p)$ and $f'(q)$ in the open interval $(p,q)$.

Suppose $f'(p) < m < f'(q)$. Set $\epsilon = \min\{m - f'(p), f'(q) - m\}$. There exists a value $\delta$ satisfying $0 < \delta < q - p$ for which, simultaneously,
$\left|\frac{f(p + \delta) - f(p)}{\delta} - f'(p)\right| < \epsilon$ and $\left|\frac{f(q) - f(q - \delta)}{\delta} - f'(q)\right| < \epsilon$.

For the function

$$g(x) = \frac{f(x + \delta) - f(x)}{\delta},$$

where $\delta$ is fixed and satisfies the preceding conditions, it follows that

$$g(p) < m < g(q - \delta).$$

The function $g(x)$ is continuous on the closed interval $[p, q - \delta]$ and therefore satisfies the intermediate value property on that interval. There must then exist a value $r$ in $(p, q - \delta)$ such that

$$g(r) = \frac{f(r + \delta) - f(r)}{\delta} = m.$$

By the Mean Value Theorem we have for some value $c$, $f(r + \delta) - f(r) = \delta f'(c)$ where $r < c < r + \delta$. It follows that $f'(c) = m$.

Alternate Solution:

Let

$$r_p(x) = \frac{f(x) - f(p)}{x - p} \qquad \text{for } x \neq p,$$

$$r_p(x) = f'(p), \qquad \text{for } x = p.$$

$r_p$ is continuous on $[p,q]$ and, by the Intermediate Value Theorem, takes on all values between $f'(p)$ and $\dfrac{f(q) - f(p)}{q - p}$. Similarly, let

$$r_q(x) = \frac{f(x) - f(q)}{x - q} \qquad \text{for } x \neq q,$$

$$r_q(x) = f'(q) \qquad \text{for } x = q.$$

$r_q$ is continuous and takes on all values between $\dfrac{f(q) - f(p)}{q - p}$ and $f'(q)$ on $[p,q]$. Now by the Mean Value Theorem, there exists a $c_p$ such that $f'(c_p) = r_p(x)$ and a $c_q$ such that $f'(c_q) = r_q(x)$ for all $x$ in $[p,q]$. Since $r_p(x)$ and $r_q(x)$ between them take on all values between $f'(p)$ and $f'(q)$, it follows by the Mean Value Theorem that $f'(x)$ takes on all values between $f'(p)$ and $f'(q)$.

21. Suppose

$$f'(x) \geq m > 0 \quad \text{and} \quad M \geq f''(x) > 0 \quad \text{on} \quad [a,b]$$

and that

$$f(r) = 0 \quad \text{where} \quad r \in [a,b].$$

Let $x_1 \in [a,b]$ and put

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

(a) Show that

$$|x_2 - r| \leq |x_1 - r|^2 \frac{M}{m}.$$

By the Mean Value Theorem, there is a $\xi$ between $x_1$ and $r$ such that

$$f'(\xi) = \frac{f(x_1) - f(r)}{x_1 - r}$$

and, since $f'(x)$ is also continuous and differentiable, there is a $\xi_1$ between $x_1$ and $\xi$ such that

$$f''(\xi_1) = \frac{f'(x_1) - f'(\xi)}{x_1 - \xi}$$

$$f'(\xi) = f'(x_1) - (x_1 - \xi)f''(\xi_1).$$

Thus

$$f(x_1) - f(r) = (x_1 - r)f'(\xi)$$
$$= (x_1 - r)[f'(x_1) - (x_1 - \xi)f''(\xi_1)].$$

And $x_2 - r = x_1 - r - \dfrac{f(x_1) - f(r)}{f'(x_1)}$ since $f(r) = 0$

$$= x_1 - r - \frac{(x_1 - r)}{f'(x_1)}[f'(x_1) - (x_1 - \xi)f''(\xi_1)]$$

$$= (x_1 - r)(x_1 - \xi)\frac{f''(\xi_1)}{f'(x_1)}.$$

Since $|x_1 - \xi| < |x_1 - r|$

$|f''(\xi_1)| \leq M$

and $|f'(x)| \geq m$

$$|x_2 - r| \leq |x_1 - r|^2 \frac{M}{m}.$$

(b) If $b - a < \frac{m}{M}k$, $0 < k < 1$, show that $|x_2 - r| \le \frac{m}{M}k^2$.

$$|x_1 - r| < |b - a| < \frac{m}{M}k$$

from (a),

$$|x_2 - r| \le \left(\frac{m}{M}k\right)^2 \frac{M}{m}$$

$$= \frac{m}{M}k^2.$$

## Solutions Exercises A7-4

1. Let $f$ be differentiable on a neighborhood of a point $a$ for which $f'(a) = 0$. If $f'(x) \leq 0$ when $x < a$ and $f'(x) \geq 0$ when $x > a$ then $f(a)$ is a minimum. If $f'(x) \geq 0$ when $x < a$ and $f'(x) \leq 0$ when $x > a$ then $f(a)$ is a maximum. Give a proof.

   We consider the case for $f(a)$ a minimum. The proof for $f(a)$ a maximum is similar. Let $x$ be a point of a deleted neighborhood of $a$. By the Mean Value Theorem there is a number $u$ such that $f(x) - f(a) = f'(u)(x - a)$ for $a < u < x$ and for $x < u < a$. From the hypothesis, whether $x < a$ or $x > a$, it follows that

   $$f'(u)(x - a) \geq 0.$$

   We conclude that $f(x) \geq f(a)$ for all $x$ in the neighborhood of $a$. Therefore $f(a)$ is a minimum.

2. Let $f$ be continuous on the closed interval $[a,b]$ and differentiable on the open interval $(a,b)$. Suppose $u$ is the one point in $(a,b)$ where $f'(u) = 0$. Prove that if $f'(x)$ reverses sign in a neighborhood of $u$ then $f(u)$ is the global extremum of $f$ on $[a,b]$ appropriate to the sense of reversal.

   By the hypothesis, $f'(x) = 0$ has only one solution, $x = u$. The derivative $f'(x)$ must have constant sign in each of the intervals $(a,u)$, $(u,b)$ or we could find another zero of the derivative by Exercises A7-3, Number 17. Since $f'(x)$ reverses sign in a neighborhood of $u$, either $f'(x) > 0$ for $x < u$, or $f'(x) > 0$ for $x > u$. We will consider the case where $f'(x) > 0$ for $x < u$ and $f'(x) < 0$ for $x > u$ (the proof for the other case is similar). Since $f'(x)$ changes sign at $u$, $f(u)$ is a maximum in a neighborhood of $u$. By the Mean Value Theorem, for the interval $[a,u]$, $f(a) - f(u) = f'(v)(a - u)$ for some $v$, $a < v < u$. Thus, in this case $f'(v)(a - u) < 0$ and $f(a) < f(u)$. In the same way, applying the Mean Value Theorem to the closed interval $[u,b]$ we can show that $f(b) < f(u)$. The only extrema on $[a,b]$ are at the endpoints or at points where $f'(x) = 0$. Since we have eliminated the endpoints as possible maxima, we conclude that $f(u)$ is a global maximum of $f$ on $[a,b]$.

3. Given a function $f$ such that $f(1) = f(2) = 4$, and such that $f''(x)$ exists and is positive throughout the interval $1 \leq x \leq 3$. What can you conclude about $f'(2.5)$? about $f(2.5)$? Prove your statements, stating whatever theorems you use in your proof. (Note: This statement of the problem differs from that in the text.)

$f(1) = f(2) = 4$.
Since $f''$ exists on the interval $[1,3]$, $f'$ is continuous and differential on $[1,3]$ and $f$ also is continuous and differentiable on $[1,3]$. By Rolle's Theorem, there is a number $u$, $1 < u < 2$, such that $f'(u) = 0$. Since $f''(x) > 0$ on the interval $[1,3]$, $f'$ is increasing on $[1,3]$ and hence, for $u < x < 3$, $f'(x) > f'(u) = 0$. Thus $f'(2.5) > 0$. Since $f'(x) > 0$ for $x$ in $(u,3)$, $f$ is increasing in $[u,3]$; since $u < 2$, it follows that $f(2.5) > f(2) = 4$.

4. Let $f$ be a differentiable function on $(a,b)$. Prove that the requirement that $f$ be increasing is equivalent to the condition that $f'(x) \geq 0$ everywhere but that every interval contains points where $f'(x) > 0$.

Let us assume first that $f$ increasing. If there were an entire interval on which $f'(x) = 0$ then by Corollary 1 to Theorem A7-4a it would follow that $f$ is constant on that interval in contradiction to the assumption that $f$ is strongly increasing. On the other hand, suppose that $f'(x) \geq 0$ but that every interval contains points where $f'(x) > 0$. Take any pair of points $x_1$, $x_2$ in $(a,b)$ with $x_1 < x_2$. We will show $f(x_1) < f(x_2)$. By hypothesis, there is a point $u$ in $(x_1,x_2)$ where $f'(u) > 0$. Since $f'(u) = \lim\limits_{x \to u} \dfrac{f(x) - f(u)}{x - u}$ it follows by Lemma 3-4 that for some sufficiently small δ-neighborhood of $u$ within $(x_1,x_2)$

$$(1) \qquad \frac{f(x) - f(u)}{x - u} > 0.$$

Choose particular values $p$, $q$ in this δ-neighborhood so that $p < u$ and $q > u$. We then have $x_1 < p < u < q < x_2$. It follows from (1) that

$$(2) \qquad f(p) < f(u) < f(q).$$

However, under the assumption $f'(x) \geq 0$ in $(a,b)$, we have from Theorem A7-4a that

$$(3) \qquad f(x_1) \leq f(p) \quad \text{and} \quad f(q) \leq f(x_2).$$

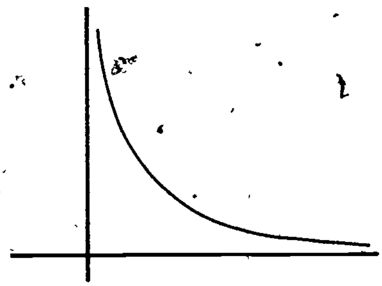
Combining the results of (2) and (3) we see that if $x_1 < x_2$ then $f(x_1) < f(x_2)$, i.e., that $f(x)$ is increasing.

5. A function $g$ is such that $g''$ is continuous and positive in the interval $(p,q)$. What is the maximum number of roots of each of the equations $g(x) = 0$ and $g'(x) = 0$, in $(p,q)$? Prove your result and give some illustrative examples.

We have $g''$ is continuous and positive on $(p,q)$. Then $g'$ is continuous and increasing on $(p,q)$ and thus can have at most one real root (else $g'(x_1) = g'(x_2) = 0$ and $g'$ is not increasing).

If $g'(x) = 0$ for any value of $x$, say $x_0$, then $g'(x) > 0$ for $x > x_0$ and $g'(x) < 0$ for $x < x_0$ ($g'$ is increasing). Then $g$ is increasing for $x > x_0$ and decreasing for $x < x_0$ and can have at most two real roots. If $g'(x) = 0$ for no value of $x$, then either $g'(x) < 0$ for all $x$ or $g'(x) > 0$ for all $x$, and $g(x)$ is either increasing or decreasing for all $x$ and can have at most one real root.

Case 1.



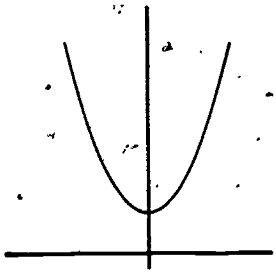
$g(x) = \frac{1}{x}$

$g''(x) > 0$

$g'(x)$ has no roots

$g(x)$ has no roots

Case 2.



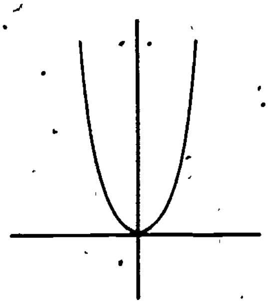
$g(x) = x^2 + 1$

$g''(x) > 0$

$g'(x)$ has no roots

$g(x)$ has no root

Case 3.



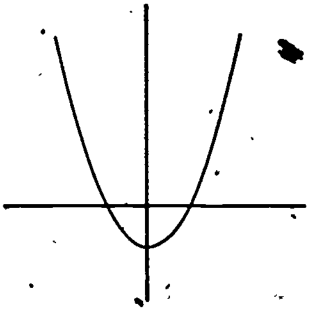
$g(x) = x^4 + x^2$

$g''(x) > 0$

$g'(x)$ has one root

$g(x)$ has one root

Case 4.



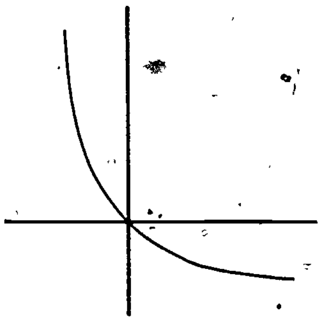
$g(x) = x^2 - 1$

$g''(x) > 0$

$g'(x)$ has one root

$g(x)$ has two roots

Case 5.



$g(x) = \frac{1}{x+1} - 1$

$g''(x) > 0$

$g'(x)$ has no roots

$g(x)$ has one root

6. Suppose that $f'(a) = f''(a) = \ldots = f^{(n-1)}(a) = 0$ but that $f^{(n)}(a) \neq 0$. Determine whether $f(a)$ is a local extremum and if it is, which kind. (Hint: consider separately the cases $n$ even and $n$ odd.)

Let $f'(a) = f''(a) = \ldots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$, then

(i) whenever $n$ is even and $f^{(n)}(a) > 0$, $f(a)$ is a local minimum value of $f$.

(ii) whenever $n$ is even and $f^{(n)}(a) < 0$, $f(a)$ is a local maximum.

(iii) whenever $n$ is odd, $f(a)$ is not a local extremum.

The proof of (i) is typical: From the proof of Theorem A7-4a and from Theorem A7-4c we know that $f^{(n-2)}(a)$ is an isolated minimum of $f^{(n-2)}$. In a deleted neighborhood of $a$ we have $f^{(n-2)}(x) > f^{(n-2)}(a) \geq 0$. We conclude that the graph of $f^{(n-4)}$ is flexed upward in the neighborhood of $a$ and hence that $f^{(n-4)}(a)$ is a minimum of $f^{(n-4)}$. Iterating this argument we obtain the desired result.

7. Let $f$ be differentiable on an interval $I$. Prove that a necessary and sufficient condition that the graph of $f$ be concave on an interval $I$ is that the slope of the chord joining a point $(x,f(x))$ to a fixed point $(a,f(a))$ is a decreasing function of $x$. (It is understood that $x$ and $a$ lie in $I$.)

<u>Note</u>: The condition is required for every point $a$ in $I$.

First we shall prove that if the graph of $f$ is concave then the slope of chords through any fixed point $a$ in $I$ is a decreasing function.

<u>Proof</u>. Let $a$ be any fixed point in $I$. Equations of chords through $(a,f(a))$ and $(q,f(q))$, for other points $q \neq a$ in $I$, will be of the form

$$y = g(x) = f(a) + (x - a)\,\frac{f(q) - f(a)}{q - a}\,.$$

The graph of $f$ is concave only if $f(p) \geq g(p)$ for all $p$ between $a$ and $q$, i.e., for all $p$ such that $a < p < q$ or $q < p < a$. In either case we have

$$f(p) \geq g(p) \geq f(a) + (x - a)\,\frac{f(q) - f(a)}{q - a}\,.$$

If $p > a$, then $q > p$, and this becomes

$$\frac{f(p) - f(a)}{p - a} \geq \frac{f(q) - f(a)}{q - a}\,.$$

If $p < a$, then $p > q$, and we have

$$\frac{f(p) - f(a)}{p - a} \leq \frac{f(q) - f(a)}{q - a}\,.$$

In both cases we have shown the slope function

$$x \to \frac{f(x) - f(a)}{x - a}$$

to be decreasing.

Now we shall prove the converse. We need to show that if the slope function for chords through any point $a$ in $I$ is decreasing, then if $y = g(x)$ is the (linear) equation of an arbitrary chord whose endpoints are $(b,f(b))$ and $(c,f(c))$, $b$ and $c$ in $I$, then $f(p) \geq g(p)$ for any $p$ between $b$ and $c$.

<u>Proof</u>. Let $b < p < c$. Then by hypothesis,

$$\frac{f(p) - f(b)}{p - b} \geq \frac{f(c) - f(b)}{c - b}\,.$$

The equation of the chord through $(b,f(b))$ and $(c,f(c))$ is

$$y = g(x) = f(b) + (x - b)\frac{f(c) - f(b)}{c - b}.$$

By definition of $g$,

$$\frac{g(x) - g(b)}{x - b} = \frac{f(c) - f(b)}{c - b},$$

so that

$$\frac{g(x) - f(b)}{x - b} \leq \frac{f(p) - f(b)}{p - b},$$

for all $x$ in I. In particular, taking $x = p$, we obtain

$$g(p) \leq f(p).$$

8. (a) Let $f$ be differentiable and its graph be concave on an interval I. Prove that the function

$$\phi(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a}, & x \neq a \\ f'(a), & x = a \end{cases}$$

is decreasing, where the fixed point $a$ is any interior point of I.

We have already shown in Number 7 that $\phi$ is decreasing on I with the point $a$ deleted. We now show that $\phi$ is decreasing on the entire interval.

Proof by Contradiction.

Suppose that $t > a$ and $\phi(a) < \phi(t)$. Then for $\epsilon = \phi(t) - \phi(a)$ there exists a $\delta > 0$ such that

$$|\phi(x) - \phi(a)| < \epsilon \text{ whenever } |x - a| < \delta.$$

Choose a value $x$ between $a$ and $t$ within the $\delta$-neighborhood of $a$. We then have $x < t$; but

$$\phi(x) - \phi(a) \leq |\phi(x) - \phi(a)| < \phi(t) - \phi(a)$$

whence $\phi(x) < \phi(t)$, in contradiction to the decreasing property of $\phi$ on the deleted interval.

A similar argument for $t < a$ completes the proof.

(b) From the result of (a), prove that a necessary and sufficient condition that the graph of $f$ be concave on I is that $f'$ be decreasing.

We first prove that if the graph of $f$ is concave on I, then $f'$ is decreasing on I.

Proof. Let $a$, $b$ be points of $I$ such that $a < b$. The function $\phi$ defined in part (a) and the function $\psi$ defined by

$$\psi(x) = \begin{cases} \dfrac{f(x) - f(b)}{x - b}, & x \neq b \\ f'(b), & x = b \end{cases}$$

are both decreasing on $I$. Since $\phi(b) = \psi(a)$, $\phi(a) \geq \phi(b)$, and $\psi(a) \geq \psi(b)$, it follows that

$$f'(a) \geq \phi(b) \geq \psi(a) \geq f'(b).$$

In other words, $f'$ is decreasing on $I$.

Conversely, we show if $f'$ is decreasing on $I$, then the graph of $f$ is flexed downward on $I$.

Proof. For any points $p, q, r$ of $I$ with $p < q < r$ we have by the Mean Value Theorem

$$\frac{f(q) - f(p)}{q - p} = f'(t), \quad \text{where} \quad p < t < q$$

and
$$\frac{f(r) - f(q)}{r - q} = f'(u), \quad \text{where} \quad q < u < r.$$

Since $t < u$, and since $f'$ is decreasing (by hyothesis) we have $f'(t) \geq f'(u)$. It follows that

$$\frac{f(q) - f(p)}{q - p} \geq \frac{f(r) - f(q)}{r - q}.$$

This inequality can be interpreted geometrically as a statement that the slope of the chord joining the fixed point $(q,f(q))$ to any point $(x,f(x))$ $x$ $I$, is a decreasing function of $x$ on $I$. It follows by Number 7 that the graph of $f$ is concave.

9. (a) Let $x$ and $y$ be two points on an interval $I$ in the domain of a function $f$. Show that a point is on the chord joining the points $(x,f(x))$ and $(y,f(y))$ on the graph of $f$ if and only if its coordinates are
$$(\theta x + (1 - \theta)y, \theta f(x) + (1 - \theta)f(y))$$
for some $\theta$ such that $0 \leq \theta \leq 1$.

If $A(a',a)$ and $B(b',b)$ are points in a plane, then $(x,y)$ is on the line $\overleftrightarrow{AB}$ if and only if

$$\frac{y - b}{x - a} = \frac{b' - b}{a' - a} \quad \text{or} \quad y = \frac{b' - b}{a' - a}(x - a) + b.$$

If we further require that $(x,y)$ be on the segment $\overline{AB}$, then $x$ must be between $a$ and $a'$, inclusive. Then $\frac{x - a}{a' - a}$ takes on all values between $0$ and $1$, inclusive. If we let $\theta = \frac{x - a}{a' - a}$

then $\quad x = (a' - a)\theta + a$

and $\quad y = (b' - b)\theta + b$

or $\quad x = \theta a' + (1 - \theta)a,$

and $\quad y = \theta b' + (1 - \theta)b,$

which was to be shown.

Since $\frac{x - a}{a' - a} = \theta$ takes on all values between 0 and 1 for $x : a < x < a'$ and $y : b < y < b'$, the converse is also true and all points $(\theta a + (1 - \theta)a', \theta b + (1 - \theta)b')$ are on $\overline{AB}$.

(b) Show that a differentiable function $f$ is convex on $I$ if and only if for all $x$ and $y$ in $I$ and all $\theta : 0 \leq \theta \leq 1$,

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y).$$

Given a function $f$ defined on an interval $I$, any point $(x,y)$ on a chord of the graph of $f$ will be of the form

$$x = \theta a + (1 - \theta)a',$$
$$y = \theta f(a) + (1 - \theta)f(a'),$$

where $\theta : 0 \leq \theta \leq 1$ and $a'$, $a$ are any two points on $I$. If we require that $f$ is convex on $I$, then, by definition, all chords lie above their corresponding arcs, or

$$f(\theta a + (1 - \theta)a') \leq \theta f(a) + (1 - \theta)f(a').$$

This equation, then, is just the analytic characterization of $f$ taken to be convex with graph flexed upward. Similarly, if the graph of $f$ is flexed downward,

$$f(\theta a + (1 - \theta)a') \geq \theta f(a) + (1 - \theta)f(a').$$

(c) Use (b) to show that the graphs of the following functions are convex.

(i) $f : x \to ax + b.$

(ii) $f : x \to x^2$

(iii) $f : x \to -\sqrt{x}.$

(i) $f(x) = ax + b$

$$\begin{aligned} f(\theta x + (1 - \theta)y) &= a(\theta x + (1 - \theta)y) + b \\ &= \theta(ax + b) + (1 - \theta)(ay + b) \\ &= \theta f(x) + (1 - \theta)f(y). \end{aligned}$$

(ii) $f(x) = x^2$

$f(\theta x + (1 - \theta)y) = (\theta x + (1 - \theta)y)^2$.

We want to show

$(\theta x + (1 - \theta)y)^2 \leq \theta x^2 + (1 - \theta)y^2$.

$$(\theta x + (1 - \theta)y)^2 - \theta x^2 - (1 - \theta)y^2$$

$$= (\theta^2 - \theta)x^2 - 2(\theta^2 - \theta)xy + ((1 - \theta)^2 - (1 - \theta))y^2$$

$$= (\theta^2 - \theta)(x - y)^2 \leq 0$$

since $(x - y)^2 > 0$ and $\theta^2 - \theta \leq 0$ $(0 \leq \theta \leq 1)$.

(iii) $f(x) = -\sqrt{x}$

$f(\theta x + (1 - \theta)y) = \sqrt{\theta x + (1 - \theta)y}$

We want to show

$\sqrt{\theta x + (1 - \theta)y} \geq \theta\sqrt{x} + (1 - \theta)\sqrt{y}$.

If this is true, then

$\theta x + (1 - \theta)y \geq \theta^2 x + 2\theta(1 - \theta)\sqrt{xy} + (1 - \theta)^2 y$.

Since both sides are nonegative, then

$(\theta - \theta^2)x - 2(\theta - \theta^2)\sqrt{xy} + (\theta - \theta^2)y \geq 0$

or $(\theta - \theta^2)(\sqrt{x} - \sqrt{y})^2 \geq 0$, which is a true statement. Starting from this fact and reversing the steps taken above, we obtain the desired result.

10. (a) Derive the following property of differentiable functions. If the graph of $f$ is concave on an interval $I$, then for all points $a$, $b$ in $I$ and any positive numbers $p$, $q$

$$f\left(\frac{pa + qb}{p + q}\right) \geq \frac{pf(a) + qf(b)}{p + q}.$$

In words, the function value of a weighted average is less than the weighted average of the function values.

We have for $f$ that

$$f(\theta a + (1 - \theta)b) \geq \theta f(a) + (1 - \theta)f(b)$$

for all $\theta : 0 \leq \theta \leq 1$.

But $\quad 0 \leq \frac{p}{p + q} \leq 1$

and $\quad 1 - \frac{p}{p + q} = \frac{q}{p + q}$.

Setting $\theta = \frac{p}{p+q}$, we have

$$f(\frac{pa}{p+q} + \frac{qb}{p+q}) = f(\frac{pa+qb}{p+q}) \geq \frac{pf(a) + qf(b)}{p+q}.$$

(b) Prove that this property is sufficient for concavity.

Since $f(\frac{pa+qb}{p+q}) \geq \frac{pf(a) + qf(b)}{p+q}$ for all $p$, $q$ positive, $\frac{p}{p+q}$ takes on all values between 0 and 1 exclusive. Then setting $\theta = \frac{p}{p+q}$, $f(\theta a + (1-\theta)b) \geq \theta f(a) + (1-\theta)f(b)$ for $\theta : 0 < \theta < 1$. For $\theta = 0$, $f(a) \geq f(a)$ and for $\theta = 1$, $f(b) \geq f(b)$. Then by Number 9, the graph of $f$ is concave. Together (a) and (b) demonstrate necessity and sufficiency. The criterion then is an alternative way of characterizing differentiable functions.

11. Prove that if $f$ is differentiable, then a necessary and sufficient condition for its graph to be concave is that

$$f(\frac{a+b}{2}) \geq \frac{f(a) + f(b)}{2}.$$

To prove necessity, use results established in Exercises A7-4, Number 10, and set $p = q = 1$.

To prove sufficiency we must show that if $f$ is continuous and

(1) $$f(\frac{a+b}{2}) \geq \frac{f(a) + f(b)}{2}$$

for all $x$, then the graph of $f$ is concave. We observe from (1) that

$$f\left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right) \geq \frac{1}{2}\left[f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_3 + x_4}{2}\right)\right]$$
$$\geq \frac{1}{4}(f(x_1) + f(x_2) + f(x_3) + f(x_4)).$$

Doubling the number of points repeatedly we obtain

(2) $$f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \geq \frac{1}{n}(f(x_1) + \dots + f(x_n))$$

where $n$ is a power of 2.

Now, if $p$ and $q$ are nonnegative integers with $p + q = n$ in (2) we obtain

(3) $$f(\frac{pa+qb}{p+q}) \geq \frac{pf(a) + qf(b)}{p+q}.$$

To prove (3) for all real values of $p$ and $q$, not just nonnegative values which add up to a power of 2, we must use the continuity of $f$. Convexity will then follow from the result of Exercise 10.

Setting $\theta = \frac{p}{p+q}$ we rewrite (3) in the form

$$(4) \qquad f(\theta a + (1-\theta)b) \geq \theta f(a) + (1-\theta)f(b)$$

where $\theta = \frac{p}{p+q}$ is any rational number with denominator a power of 2 satisfying $0 \leq \theta \leq 1$. For any value $x_0$ in $[a,b]$ we put

$$x_0 = ra + (1-r)b$$

where $r$ is now a real number satisfying $0 \leq r \leq 1$. Since $f$ is continuous at $x_0$, for any positive $\epsilon$ we may choose a $\delta > 0$ so that $|x - x_0| < \delta$ insures $|f(x) - f(x_0)| < \epsilon$. We put $x = \theta a + (1-\theta)b$ where $\theta$ is rational and has a power of 2 for its denominator. We have

$$|x - x_0| = |(\theta - r)(a - b)| = (b - a)|\theta - r| < \delta$$

provided $|\theta - r| < \frac{\delta}{b-a}$. It follows that $-\epsilon < f(x) - f(x_0) < \epsilon$. At the same time

$$\begin{aligned} |\theta f(a) + (1-\theta)f(b) - rf(a) - (1-r)f(b)| &= |\theta - r| \cdot |f(b) - f(a)| \\ &\phantom{=} \frac{\delta|f(b) - f(a)|}{b-a} \end{aligned}$$

We have

$$\begin{aligned} f(x_0) > f(x) - \epsilon &\geq \theta f(a) + (1-\theta)f(b) - \epsilon \\ &\geq rf(a) + (1-r)f(b) - \epsilon - \frac{\delta|f(b) - f(a)|}{b-a} \end{aligned}$$

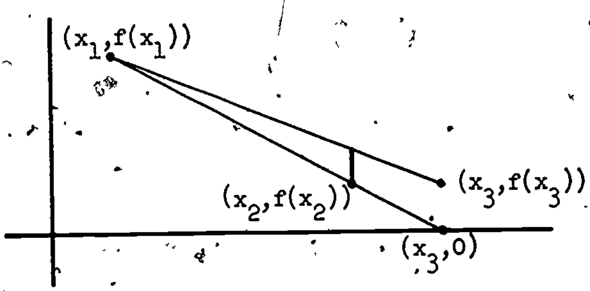
Given any $\epsilon^* > 0$, then, we take $\epsilon = \frac{\epsilon^*}{2}$ and $\delta^* \leq \delta$ but sufficiently small $\frac{\delta^*|f(b) - f(a)|}{b-a} < \frac{\epsilon^*}{2}$ and choose an approximation $\theta$ to $r$ such that $|\theta - r| < \frac{\delta^*}{b-a}$. We then have $f(ra + (1-r)b) \geq rf(a) + (1-r)f(b) - \epsilon^*$ for each positive $\epsilon^*$. This can only be true if

$$f(ra + (1-r)b) \geq rf(a) + (1-r)f(b).$$

We have extended Equation (4) to real values and the convexity of $f$ follows.

12. The graph of a differentiable function $f$ is concave and $f(x)$ is positive for all $x$. Show that $f$ is a constant function.

Solution. Suppose $f$ is nonconstant. Then there exist $x = x_1$ and $x = x_2$ such that $x_1 < x_2$ and $f(x_1) \neq f(x_2)$, say $f(x_1) > f(x_2)$. Now the line through $(x_1, f(x_1))$, $(x_2, f(x_2))$ intersects the x-axis at some point $x_3 > x_2$. We claim that $f(x_3) \leq 0$, which would contradict $f(x)$ positive. For if $f(x_3) > 0$, then the point $(x_2, f(x_2))$ would lie below the line segment connecting $(x_1, f(x_1))$ and $(x_3, f(x_3))$, contradicting $f$ concave. So the assumption of $f$ nonconstant has led to a contradiction.




13. Under what circumstances will the graph of a function $f$ and its inverse both be concave? One concave and the other convex?

Without calculus: Let $[a,b]$ be an interval in the domain of $f$. If the graph of $f$ on $[a,b]$ is concave, it is interior to the angle of $(a, f(a))$ formed by the ray to $(b, f(b))$ with the ray going vertically upward. Reflection in the line $y = x$ takes the upward ray into a ray going horizontally to the right and the other ray in the upper half-plane. If the angle was acute ($f$ increasing) the reflected graph lies in the first quadrant below the chord. If the angle was obtuse ($f$ decreasing) the reflected graph lies in the second quadrant above the chord. The same argument can be made algebraically.

With calculus: Assume $f'$ and $f''$ exist. Then $g'$ and $g''$ exist, where $g$ is the inverse of $f$. We have

$$1 = [gf(x)]' = g'f(x) \cdot f'(x)$$

or

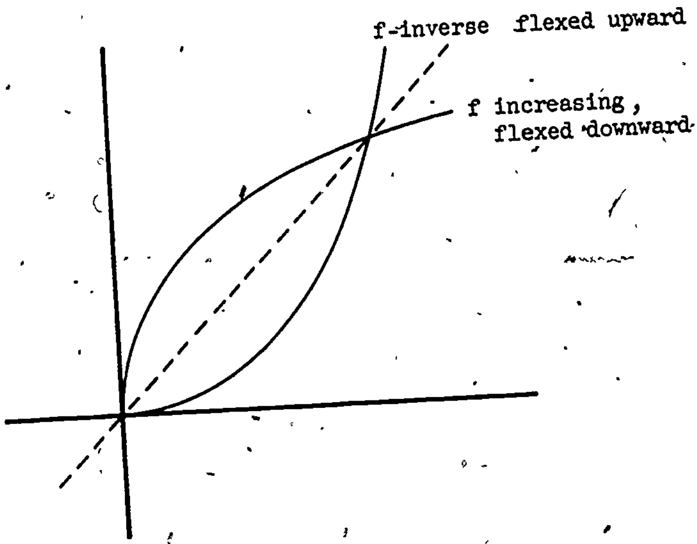
$$g'f(x) = \frac{1}{f'(x)}$$

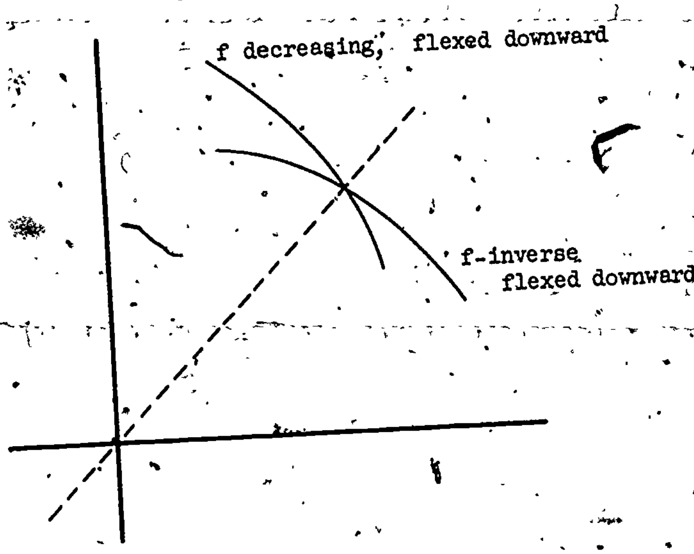
$$g''f(x) \cdot f'(x) = \frac{-f''(x)}{[f'(x)]^2}$$

or

$$g''f(x) = \frac{-f''(x)}{[f'(x)]^3}.$$

If $f''(x) < 0$, then $g''f(x) > 0$ if $f'(x) > 0$; $g''f(x) < 0$ if $f'(x) < 0$.



14. If either of $D^2xF(x)$ or $D^2F(\frac{1}{x})$ is of one sign for $x > 0$, show that the other one has the same sign. Interpret geometrically and illustrate by several examples.

Define $f$ and $g$ by $f : x \to xF(x)$ and $g : x \to F(\frac{1}{x})$. We have,

$$f''(x) = xF''(x) + 2F'(x)$$

and

$$g''(x) = \frac{F''(\frac{1}{x})}{x^4} + \frac{2F'(\frac{1}{x})}{x^3} = x^3f''(\frac{1}{x}).$$

Since $x^3 > 0$ for $x > 0$, the result follows immediately. Two routine examples follow.

(i)

$$F(x) = x^3 + x > 0$$

$$F'(x) = 3x^2 + 1 > 0$$

$$F''(x) = 6x > 0$$

$$F'(\frac{1}{x}) = \frac{3}{x^2} + 1 > 0$$

$$F''(\frac{1}{x}) = \frac{6}{x}$$

$$xF''(x) + 2F'(x) > 0 \quad \text{for all} \quad x > 0.$$

$$F''(\frac{1}{x})\left(\frac{1}{x^4}\right) + F'(\frac{1}{x})\frac{2}{x^3} > 0 \quad \text{for all} \quad x > 0.$$

(ii) $F(x) = \frac{x^4}{12} - \frac{x^3}{3} + x^2.$

$$F'(x) = \frac{x^3}{3} - x^2 + 2x; \quad F'(\frac{1}{x}) = \frac{1}{3x^3} - \frac{1}{x^2} + \frac{2}{x}$$

$$F''(x) = x^2 - 2x + 2; \quad F''(\frac{1}{x}) = \frac{1}{x^2} - \frac{2}{x} + 2.$$

$$xF''(x) + 2F'(x) = x^3 - 2x^2 + 2x + \frac{2x^3}{3} - 2x^2 + 4x$$

$$= x(\frac{5x^2}{3} - 4x + 6) > 0. \quad \text{(Discriminant} < 0)$$

$$\frac{F''(\frac{1}{x})}{x^4} + \frac{2F'(\frac{1}{x})}{x^3} = \frac{1}{x^6} - \frac{2}{x^5} + \frac{2}{x^4} + \frac{2}{3x^6} - \frac{2}{x^5} + \frac{4}{x^4}$$

$$= \frac{1}{x^6}(\frac{5}{3} - 4x + 6x^2) > 0.$$

If the graph of $x \to xF(x)$ is convex (or concave), then the graph of $x \to F(\frac{1}{x})$ is convex (or concave).

15. If the graph of $f$ is concave and $F(a) = F(b) = F(c)$ where $a < b < c$, show that $F(x)$ is constant in $(a,c)$.

If $f$ were not constant, then since the curve lies nowhere below its chord we have $f(u) < f(a)$. But $(b,f(b))$ then lies below both the chords from $(a,f(a))$ to $(u,f(u))$ and from $(u,f(u))$ to $(b,f(b))$, contradicting concavity.

16. (a) Let $a$, $b$, $c$ be three points in $I$ such that $a < b < c$, and suppose that the graph of $f$ is convex in $I$. Show that

$$f(b) \leq \frac{c - b}{c - a} f(a) + \frac{b - a}{c - a} f(c).$$

(Hint: use the result of Number 9).
Hence, show that

$$f(a) \geq \frac{c - a}{c - b} f(b) - \frac{b - a}{c - b} f(c),$$

$$f(c) \geq \frac{c - a}{b - a} f(b) - \frac{c - b}{b - a} f(a).$$

Similarly to 4(a) (this is less complicated as we are on the number line here rather than the plane), $b = \theta a + (1 - \theta)c$, for some $\theta : 0 < \theta < 1$. Then

$$\theta \cdot (a - c) = b - c$$

or

$$\theta = \frac{c - b}{c - a}, \quad (1 - \theta) = \frac{b - a}{c - a}.$$

and

$$b = \frac{c - b}{c - a} a + \frac{b - a}{c - a} c.$$

Now from 9(b), we have that

$$f(b) \leq \frac{c - b}{c - a} f(a) + \frac{b - a}{c - a} f(a),$$

for the graph of $f$ flexed upward. Multiplying both sides by $\frac{c - a}{c - b}$, we get $f(a) > \cancel{f(b)} \cdot \frac{c - a}{c - b} - f(c) \cdot \frac{b - a}{c - b}$. Similarly

$$f(c) \geq f(b) \cdot \frac{c - a}{b - a} - f(a) \cdot \frac{c - b}{b - a}.$$

(b) If the graph of $F$ is convex in a closed interval, show that $F$ is bounded in the interval.

It is an immediate consequence of the flexure of $f$ that it is bounded above. Given a closed interval $[\alpha,\beta]$ in $I$, let the chord connecting $(\alpha,f(\alpha))$, $(\beta,f(\beta))$ be described by the linear function $g$. Then $g(x) \leq \max\{f(\alpha),f(\beta)\}$ on $[\alpha,\beta]$, and by the upward flexure of $f$, $f(x) \leq g(x) \leq \max\{f(\alpha),f(\beta)\}$.

To find lower bounds for f, we use part (a). Take any point $\gamma$ in $[\alpha,\beta]$ which is not an endpoint, so that $\alpha < \gamma < \beta$. For $\alpha \le x < \gamma$, we have $x < \gamma < \beta$ and

$$f(x) \ge \frac{\beta - x}{\beta - \gamma} f(\gamma) - \frac{\gamma - x}{\beta - \gamma} f(\beta) \qquad \text{by (a),}$$

$$\ge -\left(\frac{\beta - x}{\beta - \gamma} |f(\gamma)| - \frac{\gamma - x}{\beta - \gamma} |f(\beta)|\right)$$

$$\ge -\left(\frac{\beta - \alpha}{\gamma - \alpha} |f(\gamma)|\right).$$

Similarly, for $\gamma < x \le \beta$, we have $\alpha < \gamma < x$ and using (a) again,

$$f(x) \ge -\left(\frac{\beta - \alpha}{\gamma - \alpha} |f(\gamma)|\right).$$

(c) Show by a counter example that the result in (a) is not valid for an open interval.

The graph of $f : x \to \frac{1}{x}$ is convex on $(0,x_0)$, $x_0 > 0$, but is unbounded. However, f is always bounded <u>below</u> on a finite open interval as well as finite closed interval, as can be seen from the proof of (b).

Teacher's Commentary

Appendix 8

MORE ABOUT INTEGRALS

## Solutions Exercises A8-1

1. Let $f$ be a function which takes on a maximum and minimum on every closed interval (e.g., $f$ could be a continuous function, or monotone). Let $U^*(\sigma)$ and $L^*(\sigma)$ be the upper and lower Riemann sums obtained by using the maximum and minimum values of $f(x)$ as the appropriate bounds in each interval of the subdivision.

   Let $\sigma_1$ and $\sigma_2$ be any partitions of $[a,b]$. Prove for the joint subdivision $\sigma = \sigma_1 \cup \sigma_2$ that

$$U^*(\sigma_1) \geq U^*(\sigma) \geq L^*(\sigma) \geq L^*(\sigma_2).$$

   In other terms, by adding new points to a subdivision we may reduce the difference between the upper and lower Riemann sums, and we cannot increase it.

   Let $\sigma_1 = \{x_0, x_1, \ldots, x_n\}$ and consider the partition $\tau$ of the subinterval, $\tau = \{u_0, u_1, \ldots, u_p\}$ where $u_0 = x_{k-1}$ and $u_p = x_k$, and $u_1, u_2, \ldots, u_{p-1}$ are those points (if any) of $\sigma_2$ which lie in the interior of $[x_{k-1}, x_k]$. If $M_k$ is the maximum of $f(x)$ in $[x_{k-1}, x_k]$, then for the maximum $r_j$ of $f(x)$ in any subinterval $[u_{j-1}, u_j]$ we have $r_j \leq M_k$. It follows that

$$M_k(x_k - x_{k-1}) \geq \sum_{j=1}^{p} r_j(u_j - u_{j-1}).$$

Thus we have compared the k-th term in $U^*(\sigma_1)$ with the sum of those terms in $U^*(\sigma)$ which correspond to the subintervals of $[x_{k-1}, x_k]$. It follows on addition that

$$U^*(\sigma_1) \geq U^*(\sigma).$$

In the same way, show that

$$L^*(\sigma_2) \leq L^*(\sigma).$$

Since $U^*(\sigma)$ and $L^*(\sigma)$ are upper and lower sums for the same partition we have

$$L^*(\sigma) \leq U^*(\sigma)$$

from which the result now follows.

2. Consider the function $f$ defined on $[0,1]$ by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational.} \end{cases}$$

Prove that the integral of $f$ does not exist.

Every interval contains both rational and irrational points. Consequently the maximum of $f(x)$ in every interval is 1 and the minimum is 0. For any partition $\sigma$ of $[0,1]$ we have for any upper sum $U$ and any lower sum $L$,

$$U - L = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1})$$

where $M_k \geq 1$ and $m_k \leq 0$. It follows that

$$M_k - m_k \geq 1$$

and

$$U - L \geq \sum_{k=1}^{n} (x_k - x_{k-1})$$

$$\geq 1$$

for all upper and lower sums.

The criterion of Theorem 6-3a cannot be satisfied and $f$ is not integrable over $[0,1]$.

3. Consider the function $f$ defined on $[0,1]$ by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ \frac{1}{t}, & x \text{ rational, } x = \frac{s}{t} \text{ in lowest terms.} \end{cases}$$

Prove that the integral of $f$ over $[0,1]$ exists and find its value.

As an upper bound on the number of rational points with a given denominator $t > 1$ and numerator $s$ relatively prime to $t$ and less than $t$ we have $t - 1$. For a fixed positive integer $q$ introduce the partition

$$\sigma = \{\frac{1}{2q^2}, 1 - \frac{1}{2q^2}, \frac{s}{t} \pm \frac{1}{(t-1)2q^2}, (t = 2, \ldots, q)\}.$$

Thus each point of the form $\frac{s}{t}$, $t > 1$, is contained in a subinterval of the partition of length at most $\frac{1}{(t-1)q^2}$.

Since $f(\frac{s}{t}) = \frac{1}{t} < 1$, we see that the maximum contribution to the upper sum of the rational points with denominator $t$, $(t \geq 1)$ is $\frac{1}{q^2}$. Taking the sum of the contributions of all rational points with denominators $t = 1, 2, \ldots, q$, we obtain a contribution no greater than $\frac{q}{q^2} = \frac{1}{q}$. In the remaining intervals of the subdivision we have $f(x) < \frac{1}{q}$ and since the total length of the remaining subintervals is at most 1, we have a contribution to the upper sums of no more than $\frac{1}{q}$. In this way we have found an upper sum $U$ over $\sigma$ for which $U < \frac{2}{q}$. Since we can always take the lower sum $L = 0$ and $q$ may be any positive integer whatever, it follows that the integral exists and has the value 0.

4. Give an example of a nonintegrable function $fg$ where $f$ and $g$ are each integrable.

Let $g$ be the function defined in Number 5 and take $f = \text{sgn}$. The function $fg$ is then the given function of Number 4.

## TC A8-2. The Integral of a Continuous Function

The method of establishing the existence of the integral of a continuous function given here is quite different than that given in SMSG Calculus, pp. 645ff. Our proof does not make use of the fact that a continuous function on a closed interval is uniformly continuous. By using properties of least upper and greatest lower bounds, and Theorem 8-2b we show directly that the Area Theorem holds for upper integrals, and lower integrals, then use Theorem 7-3b to establish that these are equal. Other discussions of this same technique can be found in the calculus books of Begle, Lang and Richmond.

### Solutions Exercises A8-2

1. Show that if $x \in [a,b]$ and $\delta > 0$ then $[a,b] \quad [x - \delta, x + \delta]$ is a closed interval. (Hint: Let $a_1$ be the larger of $a$ and $x - \delta$, $b_1$, the smaller of $b$ and $x + \delta$ and show that $[a_1,b_1] = [a,b] \quad [x - \delta, x + \delta]$).

   If $t \geq a_1$ then since $a_1 \geq a$ and $a_1 \geq x - \delta$ we certainly have

   $$t \geq a \quad \text{and} \quad t \geq x - \delta.$$

   Likewise, if $t \leq b_1$, then

   $$t \leq b \quad \text{and} \quad t \leq x + \delta.$$

   Thus $a_1 \leq t \leq b_1$ implies $t \in [a,b] \cap [x - \delta, x + \delta]$.

   Conversely, if $t \in [a,b] \cap [x - \delta, x + \delta]$ then $a \leq t \leq b$ and $x - \delta \leq t \leq x + \delta$ so that

   $$\max(a, x - \delta) \leq t \leq \min(b, x + \delta)$$

   and hence $t \in [a_1, b_1]$. (See Figure 1.)

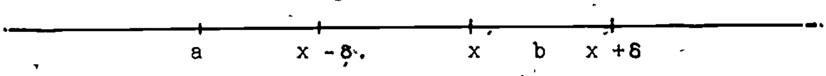



Figure 1

2. Show that if $x' > x$ and $x' \in [a,b]$, $x \in [a,b]$ then $[x,x']$ is a subinterval of $[a,b]$.

This is quite simple, for if $x \le t \le x'$ then, since $a \le x \le b$ and $a \le x' \le b$, we certainly have $a \le t \le b$. (See Figure 2.)

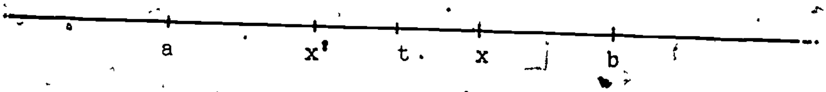



Figure 2

3. Show that

$$\int_{\underline{a}}^{b} f = -\int_{a}^{\overline{b}} (-f).$$

Suppose $\sigma = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$ and that

$$m_k \le f(x) \quad \text{for } x \in [x_k, x_{k+1}].$$

Then

$$L = \sum_{k=1}^{n} m_k(x_k - x_k - 1)$$

is a lower sum for $f$, while, since $-f(x) \le -m_k$ on $[x_k, x_{k+1}]$,

$$U = \sum_{k=1}^{n} (-m_k)(x_k - x_{k-1})$$

is an upper sum for $-f$. Since $\int_{a}^{\overline{b}} (-f)$ is the greatest lower bound of all upper sums we must have

$$\int_{a}^{\overline{b}} (-f) \le U.$$

That is,

$$-U \le -\int_{a}^{\overline{b}} (-f).$$

Since $-U = L$, this gives

$$L \le -\int_{a}^{\overline{b}} (-f)$$

so that $-\overline{\int_a^b}(-f)$ is an upper bound for the lower sums of $f$, and hence cannot be less than the least $\underline{\int_a^b} f$ of the lower sums, that is,

$$\underline{\int_a^b} f \le \overline{\int_a^b}(-f).$$

A similar argument establishes the reverse inequality.

4. Deduce from Number 3 and Theorem A8-2 that $\underline{F}' = f$ if $f$ is continuous on $[a,b]$.

Let $\overline{G}(x) = \overline{\int_a^b}(-f)$ and $\underline{F}(x) = \underline{\int_a^b} f$. Number 3 gives

$$\underline{F}(x) = -\overline{G}(x)$$

while Theorem A8-2 gives $\overline{G}' = -f$. Hence

$$D\underline{F} = D(-\overline{G}) = -D(\overline{G}) = -(-f) = f.$$

5. Show that if $f$ is continuous on $[a,b]$, then there is a number $c$ in $[a,b]$ such that

$$\int_a^b f = (b - a)f(c).$$

(Hint: Choose $c_1$ and $d_1$ in $[a,b]$ such that $f(c_1)$ and $f(d_1)$ are the respective maximum and minimum of $f$ on $[a,b]$. Show that

$$f(d_1) \le \frac{\int_a^b f}{b - a} \le f(c_1)$$

and apply the Intermediate Value Theorem.

Lemma A8-2b gives

$$f(d_1)(b - a) \le \overline{\int_a^b} f \le f(c_1)(b - a)$$

Since $f$ is continuous, $\underline{\int_a^b} f = \overline{\int_a^b} f$, so that

$$f(d_1) \le \frac{\int_a^b f}{b - a} \le f(c_1)$$

that is

$$\frac{\int_a^b f}{b - a} = d$$

lies between two values of the continuous function f. The Intermediate Value Theorem then gives a number d between $c_1$ and $d_1$ such that $f(c) = d$.

6. Use the Mean Value Theorem to show that Number 5 is true. Can you then choose c so that $a < c < b$?

Let $F(x) = \int_a^x f$, so that

$$F(a) = 0 \text{ and } F'(x) = f(x).$$

The Mean Value Theorem gives the existence of c, $a < c < b$ so that

$$\frac{F(b) - F(a)}{b - a} = F'(c)$$

that is

$$\frac{1}{b - a}\int_a^b f = f(c).$$

Thus c can be chosen in the open interval $(a,b)$.

7. Show that if f is continuous and nonnegative on $[a,b]$ with $a < b$ and if $f(x) > 0$ for some x in $[a,b]$ $\int_a^b f > 0$.
(Hint: Show that there is a $\delta > 0$ and $m > 0$ such that $f(x) \geq m$ on $[a,b] \cap [x - \delta, x + \delta]$.)

Since f is continuous at x, for $\epsilon = \frac{f(x)}{2}$ we can find $\delta_1 > 0$ such that

$$|f(t) - f(x)| < \epsilon$$

if $t \in [a,b] \cap (x - \delta_1, x + \delta_1)$, $t \neq x$, (i.e., t is in the domain of f and $0 < |t - x| < \delta_1$). Thus

$$-\epsilon < f(t) - f(x) < \epsilon$$

so, in particular

$$f(t) > f(x) - \epsilon = \frac{f(x)}{2}.$$

Take $m = \frac{f(x)}{2}$, $\delta$ any positive number $< \delta_1$. Then

$t \in [a,b] \cap [x - \delta, x + \delta]$ implies

$$f(t) \geq m.$$

Now let $[a_1,b_1] = [a,b] \cap [x - \delta, x + \delta]$ and write

$$\int_a^b f = \int_a^{a_1} f + \int_{a_1}^{b_1} f + \int_{b_1}^b f.$$

Note that

$$\int_a^{a_1} f \geq 0, \quad \int_{b_1}^b f \geq 0 \qquad \text{(since } f \geq 0\text{)},$$

and

$$\int_{a_1}^{b_1} f \geq m(b_1 - a_1) > 0 \qquad \text{(since } a_1 < b_1\text{)},$$

so that

$$\int_a^b f \geq m(b_1 - a_1) > 0.$$

8. Deduce from Number 7 that if $f'(x) > 0$ for $a < x < b$ and $f'$ is continuous on $[a,b]$ then $f$ is strictly increasing on $[a,b]$.

If $a \leq x_1 < x_2 \leq b$ then

$$\int_{x_1}^{x_2} f' = f(x_2) - f(x_1)$$

which must be positive from Number 7, that is $f(x_2) > f(x_1)$.

9. Suppose

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 2, & 1 < x \leq 2 \end{cases}$$

(a) Show directly from the definition and properties of upper integrals that:

$$\overline{F}(x) = \overline{\int_0^x} f = \begin{cases} x & , 0 \leq x \leq 1 \\ 2x - 1, & 1 < x \leq 2 \end{cases}$$

For $0 \leq x \leq 1$, we can choose the subdivision $\sigma = \{0, x\}$ of $[0,x]$ and the upper and lower sums

$$U = 1(x - 0)$$
$$L = 1(x - 0)$$

to obtain

$$x = L \le \int_0^{\overline{x}} f \le U = x$$

so that

$$\overline{F}(x) = \int_0^{\overline{x}} f = x \qquad \text{for } 0 \le x \le 1.$$

For $1 < x \le 2$, we can choose the subdivision of $[0,x]$.

$J_n = \{0, 1 - \frac{1}{n}, 1 + \frac{1}{n}, x\}$ where $n$ is any positive integer such that $1 + \frac{1}{n} < x$.

Let

$$U_n = 1[(1 - \tfrac{1}{n}) - 0] + 2[(1 + \tfrac{1}{n}) - (1 - \tfrac{1}{n})] + 2[x - (1 + \tfrac{1}{n})]$$

$$L_n = 1[(1 - \tfrac{1}{n}) - 0] + 1[(1 + \tfrac{1}{n}) - (1 - \tfrac{1}{n})] + 2[x - (1 + \tfrac{1}{n})]$$

These are then upper and lower sums for $J_n$, so that

$$L_n \le \int_0^{\overline{x}} f \le U_n.$$

Since

$$U_n = 2x - 1 + \frac{1}{n}$$

$$L_n = 2x - 1 - \frac{1}{n}$$

we have

$$2x - 1 - \frac{1}{n} \le \int_0^{\overline{x}} f \le 2x - 1 + \frac{1}{n}.$$

This holds for all $n$ sufficiently large so we conclude that

$$\overline{F}(x) = \int_0^{\overline{x}} f = 2x - 1 \quad \text{for } 1 < x \le 2.$$

(b) Does $\overline{F}$ have a derivative at $x = 1$?
Why doesn't this contradict Theorem A8-2?

$\overline{F}$ doesn't have a derivative at $x = 1$, for if $h > 0$ then $1 + h > 1$ and

$$\frac{\overline{F}(1+h) - \overline{F}(1)}{h} = \frac{(1+h) - 1}{h} = 1$$

so the right hand and left hand limits are not the same.

This doesn't contradict Theorem A8-2 as $f$ is not continuous at $x = 1$.

10. Suppose $f$ is bounded on $[a,b]$ and $\overline{F}(x) = \overline{\int_a^x} f$. Show that $\overline{F}$ is continuous on $[a,b]$. (Hint: Make use of Lemmas A8-2a, b, which hold for bounded functions.)

If $|f(x)| \leq M$, $a \leq x \leq b$, then

$$-M \leq f(x) \leq M, \quad a \leq x \leq b.$$

Therefore, if $x_1 < x_2$, apply Lemmas A8-2a and b to obtain

$$\overline{F}(x_2) - \overline{F}(x_1) = \overline{\int_a^{x_2}} f - \overline{\int_a^{x_1}} f = \overline{\int_{x_1}^{x_2}} f$$

so that

$$-M(x_2 - x_1) \leq \overline{F}(x_2) - \overline{F}(x_1) \leq M(x_2 - x_1).$$

Therefore, as $x_2 \to x_1$, $x_2 > x_1$, we have

$$\overline{F}(x_2) - \overline{F}(x_1) \to 0.$$

A similar result holds for $x_2 < x_1$ and establishes that

$$\lim_{x_2 \to x_1} \overline{F}(x_2) = \overline{F}(x_1).$$

Appendix 9

## LOGARITHM AND EXPONENTIAL FUNCTIONS AS SOLUTIONS TO DIFFERENTIAL EQUATIONS

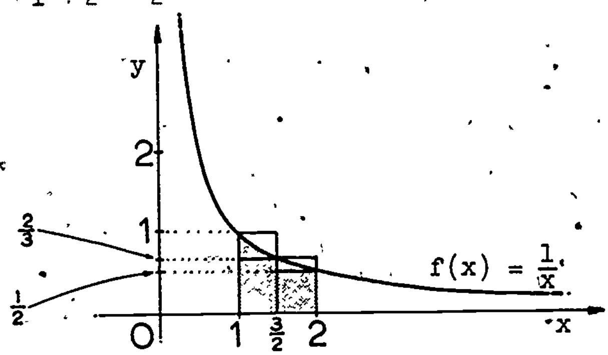
### Solutions Exercises A9-1

1. $L(2) = \int_1^2 \frac{1}{x}\,dx$

Since $\frac{1}{x} > 0$ and strictly decreasing for $x > 0$, the maximum value of $\frac{1}{x}$ on any interval $0 < a \le x \le b$ will be $\frac{1}{a}$, and the minimum value will be $\frac{1}{b}$.

Choose the partition $x_0 = 1$, $x_1 = \frac{3}{2}$, $x_2 = 2$.

$L(2)$ is the measure of the shaded area




Then the upper sum will be

$$\frac{1}{1}\left(\frac{3}{2} - 1\right) + \frac{1}{\frac{3}{2}}\left(2 - \frac{3}{2}\right) = \frac{1}{2} + \frac{1}{3}.$$

And the lower sum will be

$$\frac{1}{\frac{3}{2}}\left(\frac{3}{2} - 1\right) + \frac{1}{2}\left(2 - \frac{3}{2}\right) = \frac{1}{3} + \frac{1}{4}.$$

Thus $$\frac{1}{3} + \frac{1}{4} < L(2) < \frac{1}{2} + \frac{1}{3}.$$

2. (a) Using the arguments of Number 1 for each positive integer $k$

$$\frac{1}{k} > \frac{1}{x} > \frac{1}{k+1} \quad \text{on } k < x < k + 1.$$

And so

$$\left(\frac{1}{k+1}\right)(k + 1 - k) < \int_k^{k+1} \frac{1}{x}\,dx < \left(\frac{1}{k}\right)(k + 1 - k)$$

and

$$\sum_{k=1}^{n-1} \frac{1}{k+1} < \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x}\,dx = L(n) < \sum_{k=1}^{n-1} \frac{1}{k}.$$

Or, adding 1 to both sides, the left-hand inequality becomes

$$\sum_{k=1}^{n} \frac{1}{k} < 1 + L(n)$$

and adding $\frac{1}{n}$ to both sides of the right-hand inequality

$$\frac{1}{n} + L(N) < \sum_{k=1}^{n} \frac{1}{k}.$$

Combining, we get the required result.

(b) $\displaystyle\sum_{n=1}^{10^{100}} \frac{1}{n}$ lies between $L(10^{100}) - 1$ and $L(10^{100}) - \frac{1}{10^{100}}$, i.e.,

between $100L(10) - 1$ and approximately $100L(10)$.

Since $L(10) \sim 2.30258$ (from tables) $\displaystyle\sum_{n=1}^{10^{100}} \frac{1}{n}$ is about 230.

3. (a) On the interval $1 < x < a$

$$0 < \frac{1}{a} < \frac{1}{x} < 1$$

and $\displaystyle\int_1^a \frac{1}{a}\,dx < \int_1^a \frac{1}{x}\,dx < \int_1^a dx$

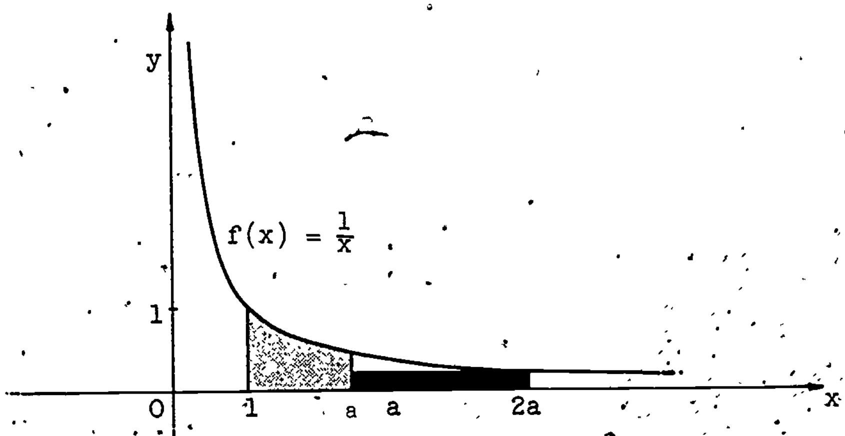
$$1 - \frac{1}{a} < L(a) < a - 1.$$

(b) $L(2a) = L(a) + \int_a^{2a} \frac{1}{x}\,dx \qquad a > 1$

$$> L(a) + \int_a^{2a} \frac{1}{2a}\,dx$$

$$= L(a) + \frac{1}{2a}(2a - a)$$

$$= L(a) + \frac{1}{2}$$




$L(a)$ is the measure of the light shaded area

$\frac{1}{2}$ is the measure of the dark shaded area

(c) $L(a) = 2L(\sqrt{a})$

$\leq 2(\sqrt{a} - 1)$ from 3(a)

$= 2\sqrt{a} - 2$

$< 2\sqrt{a}$

4. For all $x > 1$, $L(x) > 0$, so $\frac{L(x)}{x} > 0$ for any large $x$.
Also, from 3(c), for $x > 1$, $L(x) < 2\sqrt{x}$, so that $0 < \frac{L(x)}{x} < \frac{2}{\sqrt{x}}$.
Now $\lim_{x \to \infty} \frac{L(x)}{x} = 0$, $\lim_{x \to \infty} 0 = 0$ so by the Sandwich Theorem
(Appendix A6-4), $\lim_{x \to \infty} \frac{L(x)}{x} = 0$.

5. (a) $f(x) = L\left(\sqrt{\frac{x - 1}{x + 1}}\right)$

$$f'(x) = \sqrt{\frac{x + 1}{x - 1}}\,\frac{1}{2}\sqrt{\frac{x + 1}{x - 1}}\,\frac{(x + 1) - (x - 1)}{(x + 1)^2}$$

$$= \frac{1}{2}\,\frac{x + 1}{x - 1}\,\frac{2}{(x + 1)^2}$$

$$= \frac{1}{x^2 - 1}$$

or

$$f(x) = \frac{1}{2}[L(x - 1) - L(x + 1)]$$

$$f'(x) = \frac{1}{2}\left[\frac{1}{x - 1} - \frac{1}{x + 1}\right]$$

$$= \frac{1}{2}\,\frac{2}{x^2 - 1} = \frac{1}{x^2 - 1}$$

(b) $f(x) = L(x\sqrt{1 - x})$

$$f'(x) = \frac{1}{x\sqrt{1 - x}}\left(\sqrt{1 - x} - \frac{x}{2\sqrt{1 - x}}\right)$$

$$= \frac{2(1 - x) - x}{2x(1 - x)}$$

$$= \frac{2 - 3x}{2x(1 - x)}$$

or

$$f(x) = L(x) + \frac{1}{2}L(1 - x)$$

$$f'(x) = \frac{1}{x} - \frac{1}{2(1 - x)}$$

$$= \frac{2 - 3x}{2x(1 - x)}$$

(c) $f(x) = L(L(x))$

$$f'(x) = \frac{1}{xL(x)}$$

6. $f(x) = xL(x)$

$f'(x) = L(x) + 1$

$f''(x) = \frac{1}{x}$

All of these functions are defined and continuous for $x > 0$.

$f''(x) > 0$ everywhere, so $f'(x)$ is strictly increasing, and $f(x)$ is convex.

Thus, $f(x)$ will have one minimum point where $f'(x) = 0$. That is, where

$$L(x) = -1$$

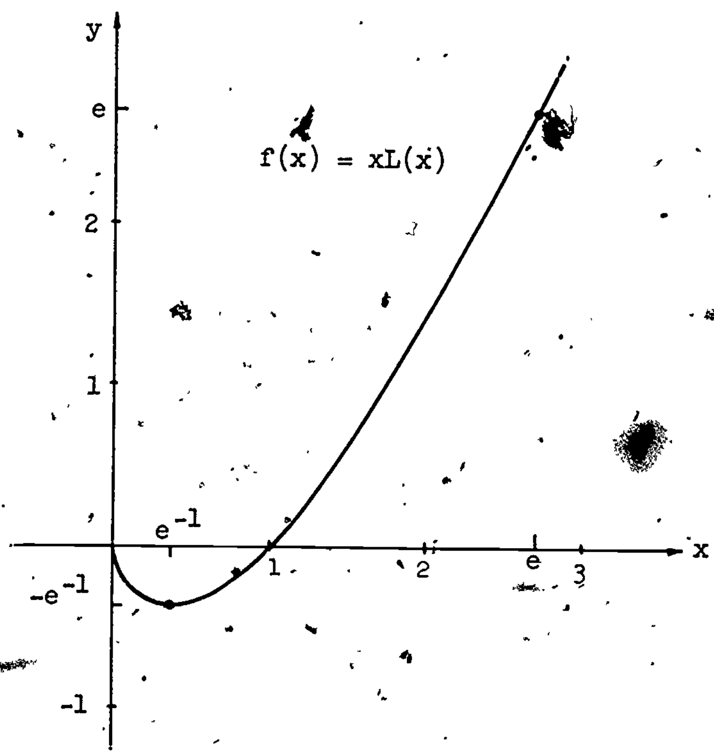
$$x = e^{-1}.$$

At this point, $f(e^{-1}) = -e^{-1}$.

As $x$ becomes large, $L(x)$ becomes large, and $xL(x)$ becomes large.

On the interval $0 < x < 1$, $x > 0$ and $L(x) < 0$, so that $xL(x) < 0$.

Also, $f(1) = 0$, $f(e) = e$.



## Solutions Exercises A9-2

1. $a^x = E(xL(a))$

(a) $f(x) = (1 - x)^x = E(xL(1 - x))$

$$f'(x) = E(xL(1-x))\left(L(1-x) - \frac{x}{1-x}\right)$$
$$= (1-x)^x \frac{(1-x)L(1-x) - x}{1-x}$$
$$= (1-x)^{x-1}[(1-x)L(1-x) - x]$$

(b) $f(x) = (L(x))^x = E(xL(L(x)))$

$$f'(x) = E(xL(L(x)))\left(L(L(x)) + \frac{1}{xL(x)}\right)$$
$$= (L(x))^x \frac{xL(x)L(L(x)) + 1}{xL(x)}$$
$$= (L(x))^{x-1} \frac{xL(x)L(L(x)) + 1}{x}$$

(c) $f(x) = x^{1/x} = E(\frac{1}{x} L(x))$

$$f'(x) = E(\tfrac{1}{x} L(x))\left(- \frac{L(x)}{x^2} + \frac{1}{x^2}\right)$$
$$= x^{\frac{1}{x} - 2}(1 - L(x))$$

2. $f(x) = x^x$ is defined for $x > 0$.

$f'(x) = x^x(L(x) + 1)$

$f''(x) = x^x(L(x) + 1)^2 + x^x \frac{1}{x}$

$f'(x)$ is everywhere increasing. So $f(x)$ is convex, and $f(x)$, has one minimum point where $f'(x) = 0$. That is where $L(x) = -1$, $x = e^{-1}$.

$$f(e^{-1}) = e^{-e^{-1}} \approx \frac{2}{3}$$

3. Let $y$ be any solution of

$$y' = cy.$$

Let $z = E(-cx)y$. Then $z' = -cE(-cx)y + E(-cx)y'$

$$= E(-cx)(y' - cy) = 0 \quad \text{for all } x.$$

Thus $z$ is some constant $K$ and $K = E(-cx)y$. Thus $y = K \cdot E(cx)$.

4. $f(x) = L(x) - 1$

$f'(x) = \frac{1}{x}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - x_n(L(x_n) - 1)$$

$$= x_n(2 - L(x_n))$$

$$x_2 = x_1(2 - L(x_1))$$

$$= 2(2 - .7)$$

$$= 2.6 \qquad L(2.6) \approx .96$$

$$x_3 = 2.6(2 - .96)$$

$$\approx 2.7$$

5. (a) $L'(1) = 1 = \lim\limits_{h \to 0} \frac{L(1 + h) - L(1)}{h} = \lim\limits_{h \to 0} L((1 + h)^{1/h})$

Now, since $L(x)$ is continuous

$$\lim_{h \to 0} L((1 + h)^{1/h}) = L(\lim_{h \to 0} (1 + h)^{1/h})$$

that is $$L(\lim_{h \to 0} (1 + h)^{1/h}) = 1$$

$$E[L(\lim_{h \to 0} (1 + h)^{1/h})] = E(1)$$

$$\lim_{h \to 0} (1 + h)^{1/h} = e$$

(b) From 5(a), take $h = \frac{1}{n}$

$$e = \lim_{\frac{1}{n} \to 0} (1 + \frac{1}{n})^n$$

$$= \lim_{n \to \infty} (1 + \frac{1}{n})^n \quad \text{since } n \to \infty \text{ as } \frac{1}{n} \to 0.$$

(c) Repeating the procedure of 5(a), we obtain

$$L'(\tfrac{1}{a}) = \lim_{h \to 0} \frac{L(\frac{1}{a} + h) - L(\frac{1}{a})}{h}$$

$$a = \lim_{h \to 0} L(1 + ah)^{1/h}$$

$$\lim_{h \to 0} (1 + ah)^{1/h} = e^a$$

and, as in 5(b), we obtain

$$\lim_{n \to \infty} (1 + \tfrac{a}{n})^n = e^a.$$

6. $A(x) = \int_0^x \frac{1}{\sqrt{1 - t^2}}\, dt, \qquad |x| < 1$

$$A'(x) = \frac{1}{\sqrt{1 - x^2}}$$

Since $A'(x)$ is everywhere positive, $A(x)$ is strictly increasing. Also $A'(x)$ exists everywhere where $|x| < 1$, so $A(x)$ is continuous.

7. (a) $A(0) = \int_0^0 \frac{1}{\sqrt{1 - t^2}}\, dt = 0$

(b) $S(A(x)) = x$

Hence $S(A(0)) = 0$, that is, $S(0) = 0$.

$$A'(x) = \frac{1}{\sqrt{1 - x^2}}$$

If $y = A(x)$, $x = S(y)$

$$S'(y) = \frac{1}{A'(x)} = \frac{1}{\frac{1}{\sqrt{1 - x^2}}} = \sqrt{1 - x^2} = \sqrt{1 - S^2(y)}.$$

$$\therefore \quad S' = \sqrt{1 - S^2}$$

(c) $S'(0) = \sqrt{1 - (S(0))^2} = 1$

(d) $S'' = \frac{-2SS'}{2\sqrt{1 - S^2}} = \frac{-S\sqrt{1 - S^2}}{\sqrt{1 - S^2}} = -S$

8. (a) $S'' + S = 0$ from 7(d)

$S''' + S' = 0$

i.e., $C'' + C = 0$

(b) Since $C = S'$, $C' = S'' = -S$ from 7(d)

(c) $C(0) = S'(0) = 1$, from 7(c)

$C'(0) = -S(0) = 0$ from 7(a) $\therefore C^2 + S^2 = 1$

(d) $C^2 = (S')^2 = 1 + S^2$ from 7(b)

9. $y = S(x)$ is a solution of $y'' + y = 0$, $y(0) = 0$, $y'(0) = 1$.

Let $z = y - S(x)$, $z(0) = 0$, $z'(0) = 0$, $z'' + z = 0$ since

$z'' + z = (y'' + y) - (S'' + S) = 0 - 0$

$$
\begin{aligned}
D((z')^2 + z^2) &= 2z'z'' + 2zz' \\
&= 2z'(z'' + z) \\
&= 0
\end{aligned}
$$

Since $D((z')^2 + z^2) = 0$ $(z')^2 + z^2 = c$

In particular, for $x = 0$

$$(z'(0))^2 + (z(0))^2 = 0 + 0 = c$$

so, for all real $x$

$$(z')^2 + z^2 = 0$$

which is true only if $z' = z = 0$.

Let $z = S(x + a) - S(x)C(a) - C(x)S(a)$

$z' = C(x + a) - C(x)C(a) + S(x)S(a)$

(since $S' = C$, and $C' = -S$).

Using, $S(0) = 0$ and $C(0) = S'(0) = 1$,

$z(0) = S(a) - 0 \cdot C(a) - 1 \cdot S(a) = 0$,

$z'(0) = C(a) - 1 \cdot C(a) + 0 \cdot S(a) = 0$.

$\therefore$ $z = 0$,

(1) $S(x + a) = S(x)C(a) + C(x)S(a)$.

Similarly,

(2) $C(x + a) = C(x)C(a) - S(x)S(a)$.

(1) and (2) are the familiar addition formulas for the sine and cosine functions.